CONCORDANCE AND ISOTOPY OF METRICS WITH POSITIVE SCALAR CURVATURE, II

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Abstract. Two positive scalar curvature metrics g_0, g_1 on a manifold M are psc-isotopic if they are homotopic through metrics of positive scalar curvature. It is well known that if metrics g_0, g_1 of positive scalar curvature on a closed compact manifold M are psc-isotopic, then they are psc-concordant: i.e., there exists a metric $\bar{g}$ of positive scalar curvature on the cylinder $M \times I$ which extends the metrics g_0 on $M \times \{0\}$ and g_1 on $M \times \{1\}$ and is a product metric near the boundary. The main result of the paper is that if psc-metrics g_0, g_1 on M are psc-concordant, then there exists a diffeomorphism $\Phi : M \times I \to M \times I$ with $\Phi|_{M\times\{0\}} = Id$ (a pseudo-isotopy) such that the metrics g_0 and $(\Phi|_{M\times\{1\}})^* g_1$ are psc-isotopic. In particular, for a simply connected manifold M with $\dim M \geq 5$, psc-metrics g_0, g_1 are psc-isotopic if and only if they are psc-concordant. To prove these results, we employ a combination of relevant methods: surgery tools related to the Gromov-Lawson construction, classic results on isotopy and pseudo-isotopy of diffeomorphisms, standard geometric analysis related to the conformal Laplacian, and the Ricci flow.

In this article, the author provides full details of the proof of the concordance/isotopy problem. The first published proof, [5], accomplished this task only partially since there was an error, see the erratum [6], which damaged the main argument of [5, Theorem 2.9], and, consequently, the proof of [5, Theorem A].

Contents

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1. Introduction

1.1. **Motivation.** Let M be a closed smooth manifold. We denote by $\mathcal{R}\mathrm{iem}(M)$ the space of all Riemannian metrics on M in the C^∞-topology, and by $\mathcal{R}\mathrm{iem}^+(M) \subset \mathcal{R}\mathrm{iem}(M)$ the subspace of metrics g with positive scalar curvature R_g. We use the abbreviation "psc-metric" for "metric with positive scalar curvature".

Throughout the article, it is assumed that M admits a psc-metric, i.e. when $\mathcal{R}\mathrm{iem}^+(M) \neq \emptyset$. It is worth to mention that the existence of psc-metrics is a well-studied question. In particular, the existence of psc-metrics is well-understood for simply-connected manifolds of dimension at least five, see [17, 35]. It is also well-known that, in general, the space $\mathcal{R}\mathrm{iem}^+(M)$ has more path-components even for simply-connected manifolds, [12, 24], see also [10] for non-simply-connected case. Furthemore, if a manifold M is spin, then the topology of the space $\mathcal{R}\mathrm{iem}^+(M)$ is at least as complicated as of the real K-theory provided $\dim M \geq 6$, see [7].

Two psc-metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ are *psc-isotopic* if there exists a *smooth* path of psc-metrics $g(t)$, $t \in I = [0,1]$, with $g(0) = g_0$ and $g(1) = g_1$. In that case, we say that the path $g(t)$ is a *psc-isotopy between* g_0 *and* g_1. In fact, psc-metrics g_0 and g_1 are psc-isotopic if and only if they belong to the same path-component in $\mathcal{R}\mathrm{iem}^+(M)$ since any continuous path of psc-metrics could be approximated by a smooth one. Let $\mathrm{Diff}(M)$ be a group of diffeomorphisms of M. The group $\mathrm{Diff}(M)$ acts on the space of metrics $\mathcal{R}\mathrm{iem}(M)$ by pull-back: $\mathrm{Diff}(M)\cdot\mathcal{R}\mathrm{iem}(M) \longrightarrow \mathcal{R}\mathrm{iem}(M)$, $(\varphi, g) \mapsto \varphi^* g$. We say that two psc-metrics g_0 and g_1 are *psc-isotopic up to a diffeomorphism* if there exists a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ and a psc-isotopy between g_0 and $\varphi^* g_1$.

Remark. We note that the term "isotopy" has several meanings in smooth topology: there is a standard term *isotopy* for two diffeomorphisms (which is equivalent to the fact that these diffeomorphisms are in the same path-component of $\mathrm{Diff}(M)$). Then there is an *isotopy group* $\mathcal{S}(M\times I)$, which consists of slice-wise diffeomorphisms $\Phi : M \times I \to M \times I$ such that $\Phi|_{M\times\{0\}} = Id_{M\times\{0\}}$. Furthermore, there is a *pseudo-isotopy group* $\mathrm{Diff}(M\times I, M\times\{0\})$ of all diffeomorphisms $\Phi : M\times I \to M\times I$ such that $\Phi|_{M\times\{0\}} = Id_{M\times\{0\}}$, see [21]. Incidentally, all these concepts turned out to be relevant to the main subject of this paper. To avoid any confusion, we use the term "psc-isotopy" and its versions for psc-metrics and their equivalence classes up to a diffeomorphism. $\diamond$

We say that two psc-metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ are *psc-concordant* if there exists a psc-metric $\bar{g}$ on $M \times I$ such that

(i) $\bar{g}|_{M\times\{0\}} = g_0$, $\bar{g}|_{M\times\{1\}} = g_1$,

(ii) $\bar{g}$ is a product-metric near the boundary $M \times \{0\} \sqcup M \times \{1\}$.

In that case, we say that the Riemannian manifold $(M \times I, \bar{g})$ is a *psc-concordance* between the psc-metrics g_0 and g_1.

Clearly, a psc-isotopy and a psc-concordance are both equivalence relations on the space $\mathcal{R}\mathrm{iem}^+(M)$ of psc-metrics. It is an easy exercise to show that any psc-isotopic metrics are psc-concordant.

From the above definitions, psc-concordance appears to be weaker than psc-isotopy. However, the difference is rather subtle and this is the main subject of this paper. In other words, we study the following

General Question: *Does psc-concordance imply psc-isotopy?*

This question is mentioned as Problem 6.3 in [29], see also [28]. As we shall see in a moment, in general, there is a potential topological obstruction for two psc-concordant metrics to be psc-isotopic: this is closely related to the obstruction which detects a gap between pseudo-isotopy and isotopy of diffeomorphisms. We conjecture that this obstruction should provide many examples of concordant psc-metrics which are not psc-isotopic. On the other hand, we give an affirmative answer to the General Question modulo of that topological obstruction: two psc-metrics are psc-concordant if and only if they are psc-isotopic up to pseudo-isotopy (see Theorem A). In particular, this implies that the answer to the General Question is always positive for simply-connected manifolds of dimension at least five (see Theorem B).

1.2. **Topological conjecture.** To identify a potential topological obstruction, we recall a few definitions and results from smooth topology. Let M be a closed compact manifold without boundary. A diffeomorphism $\Phi : M\times I \to M\times I$ is called a *pseudo-isotopy* if $\Phi|_{M\times\{0\}} = Id_{M\times\{0\}}$. Let

$$\mathrm{Diff}(M\times I, M\times\{0\}) \subset \mathrm{Diff}(M\times I)$$

be the subgroup of pseudo-isotopies. It is well-known that the group $\mathrm{Diff}(M\times I, M\times\{0\})$ acts on diffeomorphisms:

$$\mu : \mathrm{Diff}(M\times I, M\times\{0\}) \times \mathrm{Diff}(M) \longrightarrow \mathrm{Diff}(M),$$

where μ sends a pseudo-isotopy $\Phi : M\times I \to M\times I$ and a diffeomorphism $\varphi : M \to M$ to the diffeomorphism

$$\varphi\circ\left(\Phi|_{M\times\{1\}}\right) : M\to M.$$

Two diffeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$ are said to be *pseudo-isotopic* if there exists a pseudo-isotopy $\Phi\in \mathrm{Diff}(M\times I, M\times\{0\})$ such that $\mu(\Phi,\varphi_0)=\varphi_1$, i.e., $\varphi_0\circ\left(\Phi|_{M\times\{1\}}\right)=\varphi_1$. On the other hand, the group of pseudo-isotopies $\mathrm{Diff}(M\times I, M\times\{0\})$ contains a subgroup $\mathcal{S}(M\times I)$ of *isotopies*, i.e. of diffeomorphisms

$$\Phi\in \mathrm{Diff}(M\times I, M\times\{0\})$$

such that $\pi_I\circ\Phi = \pi_I$, where $\pi_I : M\times I\to I$ is the projection on the second factor. In other words, an isotopy $\Phi\in\mathcal{S}(M\times I)$ is just a smooth path of diffeomorphisms $\Phi_t : M\times\{t\}\to M\times\{t\}$ starting with the identity: $\Phi_0 = Id_{M\times\{0\}}$.

Then two diffeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$ are said to be *isotopic* if there is an isotopy $\Phi \in \mathcal{S}(M \times I)$ such that $\mu(\Phi, \varphi_0) = \varphi_1$. This is the same as a smooth path $\varphi(t)$ in the group $\mathrm{Diff}(M)$ such that $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$.

Once we identify $\mathcal{S}(M \times I)$ with the space of smooth paths in $\mathrm{Diff}(M)$ starting at the idenity, we conclude that the isotopy group $\mathcal{S}(M \times I)$ is contractible. Hence the group

$$\pi_0 \mathrm{Diff}(M \times I, M \times \{0\})$$

is the only obstruction to distinguish pseudo-isotopic and isotopic diffeomorphisms. According to J. Cerf [13], the group of path-components $\pi_0 \mathrm{Diff}(M \times I, M \times \{0\}) = 0$ for simply-connected manifolds M of dimension at least five. However, this group is non-trivial for most other manifolds, and, in general, the group of pseudo-isotopies has highly non-trivial topology.

To see a relationship to psc-metrics, consider a manifold M with

$$\pi_0 \mathrm{Diff}(M \times I, M \times \{0\}) \neq 0.$$

Assume that M admits a psc-metric g. We define a psc-metric on the cylinder $\bar{g} = g + dt^2$ on $M \times I$. Then we choose a pseudo-isotopy

$$\Phi : M \times I \to M \times I$$

which represents a nontrivial element in the obstruction group

$$\pi_0 \mathrm{Diff}(M \times I, M \times \{0\}).$$

We equip the cylinder $M \times I$ with the psc-metric $\Phi^* \bar{g}$. By construction, the metrics $g_0 = g$ and $g_1 = (\Phi|_{M \times \{1\}})^* g$ are psc-concordant. The question of whether the metrics $g_0 = g$ and g_1 are psc-isotopic or not is open (provided that the diffeomorphism $\Phi|_{M \times \{1\}}$ is not isotopic to the identity).

Topological Conjecture. *Let $\Phi \in \pi_0 \mathrm{Diff}(M \times I, M \times \{0\})$ be a nontrivial element such that $\Phi|_{M \times \{1\}}$ is not isotopic to the identity. Then the metrics g_0 and $g_1 = (\Phi|_{M \times \{1\}})^* g$ are not psc-isotopic.*

Remark. W. Steimle has communicated to the author the following result: there exist many non-trivial concordances $\Phi \in \mathrm{Diff}(M \times I, M \times \{0\})$ such that $\Phi|_{M \times \{1\}} = Id_M$, see [31, Theorem 1.2]. Thus the condition that $\Phi|_{M \times \{1\}}$ is not isotopic to the identity is essential in the above conjecture. The author is grateful to W. Steimle for clarifying this issue.

It is worth noting here that the obstruction group $\pi_0 \mathrm{Diff}(M \times I, M \times \{0\})$ is often non-trivial; for instance, the obstruction group is "almost always" non-zero if the fundamental group $\pi_1 M$ contains torsion (see, say, [23, 27] for more details). $\diamond$

Remark. In dimension four, D. Ruberman [30] constructed examples of simply connected manifolds M^4 and psc-concordant psc-metrics g_0 and g_1 which are not psc-isotopic. In that case, the obstruction comes from the Seiberg-Witten invariant and again, it is topological by nature: it detects

the gap between an isotopy and a pseudo-isotopy of diffeomorphisms for 4-manifolds. In particular, those examples of psc-metrics are psc-isotopic up to pseudo-isotopy. In particular, the counterexample psc-metrics g_0 and g_1 constructed in [30] both project to the same path-component of the moduli space $\mathcal{R}\mathrm{iem}^+(M)/\mathrm{Diff}(M)$ of psc-metrics (or its version, see [9]). In other words, the above potential and actual examples of psc-concordant metrics g_0 and g_1 which are not psc-isotopic in the space $\mathcal{R}\mathrm{iem}^+(M)$ are still homotopic in the moduli space $\mathcal{R}\mathrm{iem}^+(M)/\mathrm{Diff}(M)$ of psc-metrics. $\diamond$

1.3. **Algorithmic unsolvability.** There is another important aspect concerning the above General Question. Let $(M \times I, \bar{g})$ be a psc-concordance between psc-metrics g_0 and g_1. If we think about

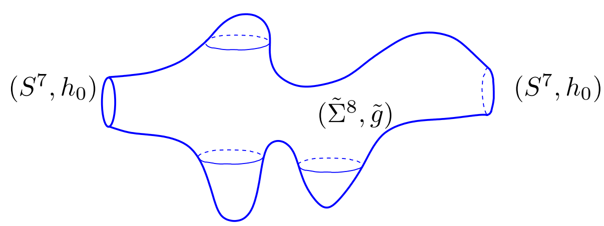

FIGURE 1. An "exotic" psc-concordance

the cylinder $(M \times I, \bar{g})$ isometrically imbedded into Euclidian space, then it might be extremely long and could contain very complicated features which cannot be effectively described analytically or topologically. In dealing with these issues, it is important to keep in mind the following result:

Theorem 1.1. (M. Gromov) *The problem of deciding whether two psc-concordant psc-metrics are psc-isotopic is algorithmically unsolvable.*

Proof. (Sketch) The proof of Theorem 1.1 is based on a well-known fact, namely, that the problem of recognizing the trivial group out of given finite sets of generators and relations is algorithmically unsolvable. To get to a psc-concordance, we take finite "unrecognizable" sets of generators and relations; this gives us finite 2-complex K, which has a unique zero cell, as many 1-cells as the number of generators, and with 2-cells attached according to the relations. By construction, $\pi_1 K = 0$. We embed K into the Euclidean space $\mathbf{R}^5$ and denote by $T(K)$ its closed tubular neighbourhood in $\mathbf{R}^5$. Then we double $T(K)$ to form a closed, simply connected compact manifold

$$X^5 = T(K) \cup_{\partial T(K)} -T(K).$$

The product $X^5 \times S^3$ has an obvious psc-metric. By construction, the manifold $X^5 \times S^3$ is simply-connected, and there is a surgery (of an appropriate codimension) to turn $X^5 \times S^3$ into a homotopy sphere Σ^8 equipped with a psc-metric. Then, after deleting two small disks, one constructs an *exotic* psc-concordance $(\tilde{\Sigma}^8, \tilde{g})$ between two round standard spheres (S^7, h_0), see Fig. 1. It is indeed exotic since there is no algorithm which would turn that psc-concordance into psc-isotopy: otherwise, it would recognize along the way that the original system of generators and relations determines a trivial group. $\square$

In particular, Theorem 1.1 implies that in order to make any progress on whether a psc-concordance implies a psc-isotopy, we have to employ some tools which are "non-algorithmic" by their nature, such as surgery.

1.4. **Main results.** Since in general, there are topological obstructions to finding a psc-isotopy for psc-concordant metrics, we would like to separate the *geometric* issues from the *topological* ones concerning the problem of whether psc-concordance implies psc-isotopy. Here is the first main result:

Theorem A. *Let M be a closed compact manifold with $\dim M \geq 3$. Then, for any two psc-concordant metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ there exists a pseudo-isotopy*

$$\Phi \in \mathrm{Diff}(M \times I, M \times \{0\})$$

such that the psc-metrics g_0 and $\Phi^* g_1 = (\Phi|_{M\times\{1\}})^* g_1$ are psc-isotopic.

According to J. Cerf's result [13], $\pi_0 \mathrm{Diff}(M \times I, M \times \{0\}) = 0$ for any simply connected manifold M with $\dim M \geq 5$. Hence, in that case, there is no obstruction for two pseudo-isotopic diffeomorphisms to be isotopic. This gives the second main result as a corollary of Theorem A.

Theorem B. *Let M be a closed simply connected manifold with $\dim M \geq 5$. Then two psc-metrics g_0 and g_1 on M are psc-isotopic if and only if the metrics g_0, g_1 are psc-concordant.*

2. The strategy to prove Theorem A

2.1. **First steps.** First, we would like to specify the statement of Theorem A for a given compact closed manifold M. We use the abbreviation "$(\mathbf{C}\Longleftrightarrow\mathbf{I})(M)$" for the following statement:

- Let $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ be any psc-concordant metrics. Then, there exists a pseudo-isotopy $\Phi \in \mathrm{Diff}(M \times I, M \times \{0\})$ and a psc-concordance $\bar{g}$ of g_0 and g_1 such that the psc-metrics g_0 and $\Phi^* g_1 = (\Phi^* \bar{g})|_{M\times\{1\}}$ are psc-isotopic.

It turns out that it is much easier to prove the statement $(\mathbf{C}\Longleftrightarrow\mathbf{I})(M)$ if the manifold M does not admit any Ricci-flat metric. To reduce Theorem A to such a case, we have to make two more steps as follows.

2.2. **PSC-concordance-isotopy surgery Theorem.** Let M be a closed manifold, $\dim M = n-1$, and $S^p \subset M$ be an embedded sphere in M with trivial normal bundle. We assume that it is embedded together with its tubular neighbourhood $S^p \times D^{q+1} \subset M$. Here $p + q + 1 = n - 1$. Then we denote by M' the manifold which is resulting from the surgery along the sphere S^p:

$$M' = (M \setminus (S^p \times D^{q+1})) \cup_{S^p \times S^q} (D^{p+1} \times S^q).$$

The codimension of the sphere $S^p \subset M$ is called the *codimension of the surgery*. In the above terms, the codimension of the above surgery is $(q + 1)$.

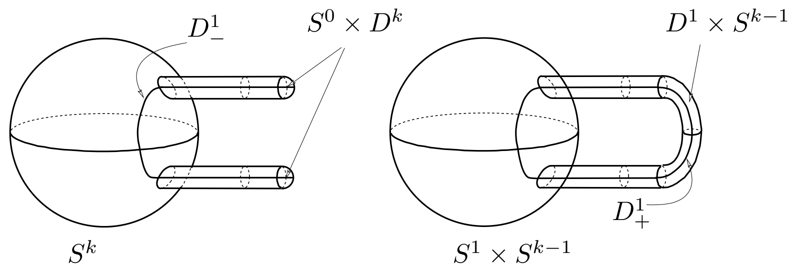

FIGURE 2. The surgery $S^k \Longrightarrow S^1 \times S^{k-1}$.

Example. Let $M = S^k$, $k \geq 4$. Then, it is easy to make a surgery of codimension k on S^k to construct $M' = S^1 \times S^{k-1}$, see Fig. 2. On the other hand, there is a second surgery which recovers S^k from $S^1 \times S^{k-1}$, see Fig. 3. We notice that both surgeries are of codimension at least three, provided $k \geq 4$. There is more general construction: a surgery along a submanifold $\Sigma \subset M$ which

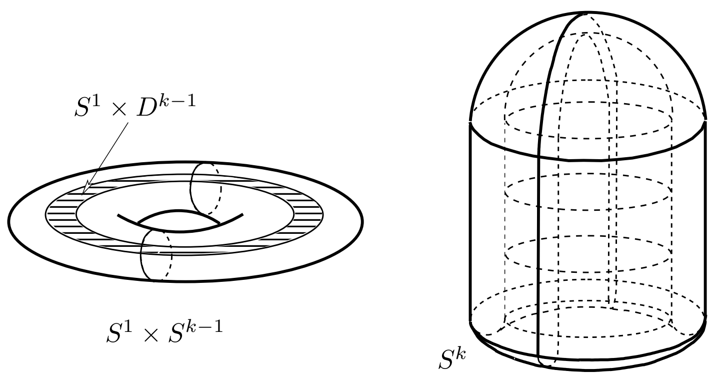

FIGURE 3. The surgery $S^1 \times S^{k-1} \Longrightarrow S^k$.

is embedded into M together with a trivial normal bundle, i.e. $\Sigma \times D^{q+1} \subset M$. We assume that $\Sigma = \partial X$. Then we form a new manifold:

$$M'_{\Sigma,X} = \big(M \setminus (\Sigma \times D^{q+1})\big) \cup_{\Sigma \times S^q} (X \times S^q).$$

Then we say that the manifold $M'_{\Sigma,X}$ is constructed out of M by a surgery of codimension $q+1$.

Example. For instance, there is a surgery

$$S^1 \times S^{k-1} \Longrightarrow S^1 \times S^1 \times S^{k-2},$$

along $\Sigma = S^1 \times S^0$, where $\Sigma = \partial(S^1 \times (D^1 \times S^{k-2})$, such that:

$$\begin{aligned} S^1 \times S^1 \times S^{k-2} &= \Big((S^1 \times S^{k-1}) \setminus (S^1 \times S^0 \times D^{k-1})\Big) \cup_{S^1 \times S^0 \times S^{k-2}} (S^1 \times (D^1 \times S^{k-2})) \\ &= S^1 \times \Big((S^{k-1} \setminus (S^0 \times D^{k-1})) \cup_{S^0 \times S^{k-2}} (D^1 \times S^{k-2})\Big). \end{aligned}$$

Similarly, there is a surgery

$$S^1 \times S^1 \times S^{k-2} \Longrightarrow S^1 \times S^{k-1}$$

along $\Sigma' = S^1 \times S^1$ with $X = S^1 \times D^2$.

Definition 2.1. In the case if $\Sigma = S^p$ and $X = D^{p+1}$ or, respectively, $\Sigma = S^k \times S^{p-k}$ and $X = S^k \times D^{p-k+1}$, we say that the surgery along Σ is *spherical* or, respectively, *almost spherical.*

Definition 2.2. Let M and M' be manifolds such that:

- M' can be constructed out of M by a finite sequence of spherical or almost spherical surgeries of codimension at least three, and
- M can be constructed out of M' by a finite sequence of spherical or almost spherical surgeries of codimension at least three.

Then, we say that *M and M' are related by admissible surgeries.*

Remark. In particular, the manifolds S^k and $T^{k-3}\times S^3$ are related by admissible surgeries if $k\geq 4$. Moreover, the manifolds

$$M \cong M\#S^k \quad \text{and} \quad M' = M\#(T^{k-3}\times S^3)$$

are also related by admissible surgeries. $\diamond$

We prove the following result in Section 12:

Theorem 2.3. *Let M and M' be closed manifolds which are related by admissible surgeries. Then the statements $(\mathbf{C}\Longleftrightarrow\mathbf{I})(M)$ and $(\mathbf{C}\Longleftrightarrow\mathbf{I})(M')$ are equivalent.*

In particular, Theorem 2.3 implies the following result:

Corollary 2.4. *Let M be a closed manifold with $\dim M = k\geq 4$. We let $M' := M\#(T^{k-3}\times S^3)$. Then the statements $(\mathbf{C}\Longleftrightarrow\mathbf{I})(M)$ and $(\mathbf{C}\Longleftrightarrow\mathbf{I})(M')$ are equivalent.*

2.3. **Surgery and Ricci-flatness.** As it turns out, it is easy to use surgery in order to construct a manifold which does not admit any Ricci-flat metric. The following result follows directly from [14, Theorem 3]:

Theorem 2.5. *Let M be a closed connected manifold with $\dim M = k\geq 4$. Then the manifold*

$$M' = M\#(S^3\times T^{k-3})$$

does not admit a Ricci-flat metric.

Corollary 2.4 and Theorem 2.5 imply that it is enough to prove Theorem A under the restriction that a manifold M does not admit a Ricci-flat metric.

2.4. **The simplest case when psc-concordance implies psc-isotopy.** Now we may return to Theorem A. We start with a psc-concordance $(M \times I, \bar{g})$. Let

$$\pi_M : M \times I \to M, \quad \pi_I : M \times I \to I \tag{2.1}$$

be projections on the first and the second factors. This gives us a coordinate system (x,t) on the product $M \times I$.

We assume that the metric $\bar{g}$ is given as $\bar{g} = g_t + dt^2$ with respect to this coordinate system. Here $g_t = \bar{g}|_{M\times\{t\}}$. Moreover, we assume that the mean curvature H_{g_t} along the hypersurface $M \times \{t\}$ is identically zero for each $t \in I$.

As it turns out, this is an ideal situation which guarantees that the metrics g_t have positive scalar curvature for all $t \in I$. Indeed, if $\bar{g} = g_t + dt^2$, then the Gauss formula could be written as follows:

$$R_{\bar{g}} = R_{g_t} + 2\partial_0 H_t - H_t^2 - |A_t|^2, \tag{2.2}$$

where A_t is the second fundamental form of the hypersurface $M \times \{t\}$, H_t is the mean curvature along $M \times \{t\}$, and $\partial_0 H_t$ its derivative in the t-direction. Thus if the hypersurfaces $M \times \{t\}$ are minimal for all $t \in I$ (i.e. $H_t \equiv 0$), then (2.2) implies

$$R_{\bar{g}} = R_{g_t} - |A_t|^2.$$

Hence $R_{g_t} > 0$ for all $t \in I$ if $R_{\bar{g}} > 0$. We summarize these observations:

Proposition 2.6. *Let $(M \times I, \bar{g})$ be a Riemannian manifold such that*

- **(a)** $R_{\bar{g}} > 0$*;*
- **(b)** $\bar{g} = g_t + dt^2$ *with respect to the coordinate system given by* (2.1)*;*
- **(c)** $H_{g_t} \equiv 0$ *for all* $t \in I$*.*

Then the metrics g_t have positive scalar curvature for all $t \in I$. In particular, the family of psc-metrics $\{g_t\}$ provides a psc-isotopy between g_0 and g_1.

Clearly the conditions **(a)**, **(b)** and **(c)** are too strong to expect that for given psc-metrics g_0, g_1 on M, one can easily find a psc-concordance $(M \times I, \bar{g})$ like that. Moreover, it is very difficult to balance the conditions **(a)**, **(b)** and **(c)**. For example, a small conformal change of the metric $\bar{g}$ maintains positivity of the scalar curvature, but it easily violates both of the conditions **(b)** and **(c)**.

2.5. **Slicing functions.** Now we are getting close to a central problem here: for a given psc-concordance $(M \times I, \bar{g})$ between psc-metrics g_0 and g_1, we should look for a *slicing function* $\bar{\alpha} : M \times I \to I$ such that the curve of Riemannian manifolds (M_t, g_t) provides a desired psc-isotopy. Here $M_t = \bar{\alpha}^{-1}(t)$, and $g_t = \bar{g}|_{M_t}$.

Let M be a closed smooth manifold. We consider the direct product $M\times I$ and the projection $\pi_I : M\times I\to I$ on the second factor.

Definition 2.7. A slicing function $\bar{\alpha} : M\times I\to I$ is a smooth function such that

- **(i)** it has no critical points;
- **(ii)** it agrees with the projection $\pi_I : M\times I\to I$ near the boundary
$$\partial(M\times I)=M\times\{0\}\sqcup M\times\{1\},$$
in particular, $\bar{\alpha}^{-1}(0)=M\times\{0\}$ and $\bar{\alpha}^{-1}(1)=M\times\{1\}$.

We denote by $\mathcal{E}(M\times I)$ the space of slicing functions with the Whitney topology. We will review necessary relults on the space of slicing functions in the next section.

2.6. **Sufficient conditions.** Let $\dim M=n-1\geq 4$. Let $\bar{C}$ be a conformal class of metrics on $M\times I$, and $C_0=\bar{C}|_{M\times\{0\}}$ and $C_1=\bar{C}|_{M\times\{1\}}$. We say that a conformal class C on M is *positive* if it contains a psc-metric. Then we say that $(M\times I,\bar{C})$ is a *conformal psc-concordance between positive conformal classes C_0 and C_1* if there exists a psc-metric $\bar{g}\in\bar{C}$ with zero mean curvature along the boundary. As it turns out, conformal psc-concordance is equivalent to psc-concordance (see Theorem 3.2 and Section 3 for more details).

Given a conformal psc-concordance $(M\times I,\bar{C})$, we choose a metric $\bar{g}\in\bar{C}$ with minimal boundary condition, which does not necesarily have positive scalar curvature. Next, we choose a slicing function $\bar{\alpha} : M\times I\to I$, $\bar{\alpha}\in\mathcal{E}(M\times I)$. In particular, the slicing function $\bar{\alpha}$ gives the coordinates (x,t) on $M\times I$. Then for each $t<t'$, we define a manifold $\bar{M}^*_{t,t'}=\bar{\alpha}^{-1}([t,t'])$ equipped with a metric $\bar{g}^*_{t,t'}=\bar{g}|_{\bar{M}^*_{t,t'}}$.

Furthemore, for each t,t', $0\leq t<t'\leq 1$, we let $\xi_{t,t'} : [0,1]\to[t,t']$ be a linear function sending $\tau\mapsto(1-\tau)t+\tau t'$. This gives a diffeomorphism

$$\bar{\xi}_{t,t'} : M\times[0,1]\to\bar{M}^*_{t,t'},\qquad (x,\tau)\mapsto(x,\xi_{t,t'}(\tau)). \tag{2.3}$$

We use the map $\bar{\xi}_{t,t'}$ to stretch the manifold $(\bar{M}^*_{t,t'},\bar{g}^*_{t,t'})$ in the horizontal direction, and denote by $(\bar{M}_{t,t'},\bar{g}_{t,t'})$ the resulting stretched manifold, $\bar{M}_{t,t'}=M\times I$, see Fig. 4. Then $\partial\bar{M}_{t,t'}=M_t\sqcup M_{t'}$, where $M_\tau=\bar{\alpha}^{-1}(\tau)$, $\tau=t,t'$. Let $A_{\bar{g}_{t,t'}}$ be the second fundamental form of the metric $\bar{g}_{t,t'}$ along

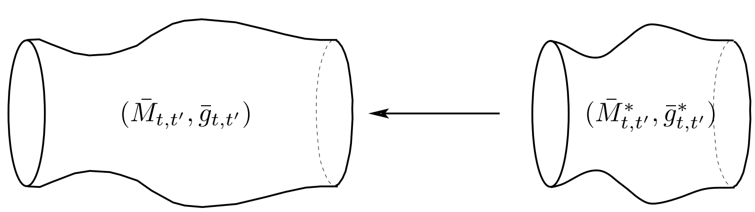

FIGURE 4. Stretching $(\bar{M}^*_{t,t'},\bar{g}^*_{t,t'})$ to get $(\bar{M}_{t,t'},\bar{g}_{t,t'})$.

the boundary. We denote by $h_{\bar{g}_{t,t'}} = \frac{1}{n-1} \operatorname{tr} A_{\bar{g}_{t,t'}}$ the normalized mean curvature along the boundary $\partial\bar{M}_{t,t'} = M_t \sqcup M_{t'}$. Let

$$L_{\bar{g}_{t,t'}} = a_n \Delta_{\bar{g}_{t,t'}} + R_{\bar{g}_{t,t'}}, \quad \text{where} \quad a_n = \tfrac{4(n-1)}{n-2},$$

be the conformal Laplacian on the manifold $(\bar{M}_{t,t'}, \bar{g}_{t,t'})$.

We denote by $\lambda_1(L_{\bar{g}_{t,t'}})$ the principal eigenvalue of the minimal boundary problem:

$$\text{(2.4)} \qquad \left\{ \begin{array}{lcll} L_{\bar{g}_{t,t'}} u & = & a_n \Delta_{\bar{g}_{t,t'}} u + R_{\bar{g}_{t,t'}} u = \lambda_1(L_{\bar{g}_{t,t'}}) u & \text{on } \bar{M}_{t,t'} \\ & & & \\ B_{\bar{g}_{t,t'}} u & = & \partial_\nu u + \frac{n-2}{2} h_{\bar{g}_{t,t'}} u = 0 & \text{on } \partial\bar{M}_{t,t'}. \end{array} \right.$$

Here ∂_ν is the outward unit normal vector field along the boundary. Let T be the triangle

$$T = \{ \ (t,t') \mid t \leq t' \ \} \subset \mathbf{R}^2,$$

where we give $\mathbf{R}^2$ the coordinates (t,t'). It turns out, we obtain a continuous function

$$\Lambda_{(M\times I, \bar{g}, \bar{\alpha})} : T \to \mathbf{R}, \quad \Lambda_{(M\times I, \bar{g}, \bar{\alpha})} : (t,t') \mapsto \lambda_1(L_{\bar{g}_{t,t'}}).$$

We give more details on that in Section 4.7. Now the idea is to replace the sufficient conditons **(a)**, **(b)** and **(c)** from Proposition 2.6 with the non-negativity of the function $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})}$. It turns out this is enough provided the manifold M does not admit a Ricci-flat metric.

Theorem 2.8. *Let M be a closed manifold with $\dim M = n-1 \geq 3$ which does not admit a Ricci-flat metric. Assume that $(M \times I, \bar{C})$ is a conformal psc-concordance between positive conformal classes C_0 and C_1, $\bar{g} \in \bar{C}$ is a metric with zero mean curvature along the boundary, and $\bar{\alpha} : M \times I \to I$ is a slicing function such that $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})} \geq 0$. Then any two psc-metrics $g_0 \in C_0$ and $g_1 \in C_1$ are psc-isotopic up to pseudo-isotopy, i.e., there exists a pseudoisotopy*

$$\Phi : M \times I \to M \times I$$

such that the psc-metrics g_0 and $(\Phi|_{M\times\{1\}})^* g_1$ are psc-isotopic.

2.7. **Comments on Theorem 2.8.** We would like to answer the following obvious questions:

(1) Why do we need the condition that M does not admit a Ricci-flat metric?

(2) How does a pseudo-isotopy appear here?

(1) Assume the slicing function $\bar{\alpha}$ coincides with the projection $\pi_I : M \times I \to I$. Furthemore, we assume that the metric $\bar{g}$ is given as $\bar{g} = g_\tau + d\tau^2$. Consider now the conformal Laplacian $L_{\bar{g}_{t,t'}}$ on $\bar{M}_{t,t'}$ with the minimal boundary condition. Assuming that $\Lambda(t,s) = \lambda_1(L_{\bar{g}_{t,s}}) \geq 0$ for all pairs $t < s$, one can show that the conformal Laplacian L_{g_t} on the slice (M_t, g_t) has nonnegative principal eigenvalue $\lambda_1(L_{g_t}) \geq 0$ for each t.

Then we find positive eigenfunctions $u(t)$ corresponding to the eigenvalue $\lambda_1(L_{g_t})$; the functions $u(t)$ depend continuously on t. We make a slice-wise conformal deformation $\hat{g}_t = u(t)^{\frac{4}{n-3}} g_t$, then

$$R_{\hat{g}_t} = u(t)^{-\frac{4}{n-3}} \lambda_1(L_{g_t}) = \begin{cases} > 0 & \text{if } \lambda_1(L_{g_t}) > 0, \\ \equiv 0 & \text{if } \lambda_1(L_{g_t}) = 0. \end{cases}$$

If $\lambda_1(L_{g_t}) > 0$ for all t, then we have obtained a path of psc-metrics.

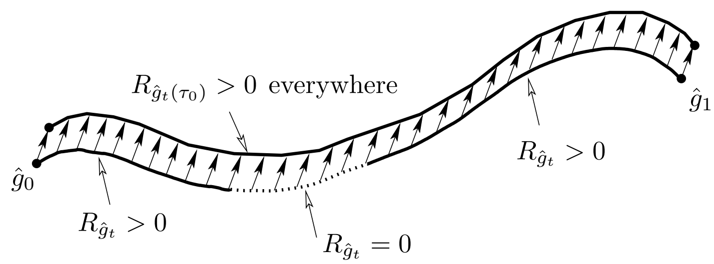

FIGURE 5. Ricci flow applied to the path $\hat{g}_t$.

If $\lambda_1(L_{g_t}) = 0$, we require the condition that M does not admit a Ricci flat metric. In that case, if the metric $\hat{g}_t$ is scalar flat, it cannot be Ricci-flat. Thus for each t we can start the Ricci flow $\hat{g}_t(\tau)$ with the initial metric $\hat{g}_t(0) = \hat{g}_t$, so that short-time existence of the Ricci Flow yields a path of psc-metrics $\hat{g}_t(\tau_0)$, where the parameter τ_0 is small (see Fig. 5), see, for instance, [36, Proposition 2.5.4 and Theorem 5.2.1]. $\diamond$

(2) Now we let $\bar{\alpha}$ be an arbitrary slicing function, but we assume that the metric $\bar{g}$ and the function $\bar{\alpha}$ are coupled as follows. First, we assume that $|\nabla\bar{\alpha}|_{\bar{g}} = 1$. We consider a trajectory γ_x of the gradient vector field $\nabla\bar{\alpha}$ satisfying the initial condition $\gamma_x(0) = x$, $x \in M \times \{0\}$. This generates a pseudo-isotopy $\Phi : M \times I \to M \times I$ given by the formula

$$\Phi : (x,t) \mapsto (\pi_M(\gamma_x(t)), \pi_I(\gamma_x(t)) := (y,s).$$

Then we obtain a metric $\tilde{g} = (\Phi^{-1})^*(\bar{g}) = g_s + ds^2$. Thus, the condition $|\nabla\bar{\alpha}|_{\bar{g}} = 1$ converts to the condition **(c)**. Then one can generalize the above argument we use in **(1)** to show that g_0 and g_1 are isotopic up to pseudo-isotopy. $\diamond$

2.8. **Necessary condition.** Here is the necessary condition:

Theorem 2.9. *Let M be a closed manifold with $\dim M = n-1 \geq 3$, and $C_0, C_1 \in \mathcal{C}(M)$ be conformaly psc-concordant conformal classes. Then there exist*

- **(i)** *a conformal psc-concordance $(M \times I, \bar{C})$ between C_0 and C_1,*
- **(ii)** *a metric $\bar{g} \in \bar{C}$ with minimal boundary condition,*
- **(iii)** *a slicing function $\bar{\alpha} \in \mathcal{E}(M \times I)$*

such that $\Lambda_{(M\times I,\bar{g},\bar{\alpha})} \geq 0$.

Remark. It is easy to see that Theorem 2.9 together with Theorem 2.8 (taking into account Corollary 2.4 and Theorem 2.5) imply Theorem A. ◇

Remark. We note that in dimension three the existence of a positive scalar curvature metric prevents the existence of a Ricci-flat metric. Indeed, in dimension three Ricci-flat implies sectional-flat. Then, according to [18, Corollary C], such manifold cannot carry positive scalar curvature metric. Hence, the hypothesis of $\dim M \geq 4$ in Corollary 2.4 and Theorem 2.5, in proving a result where $\dim M \geq 3$, is not a problem. Furthermore, we notice that Theorem A holds if $\dim M = 3$ for trivial reasons, since in that case the space $\mathcal{R}\mathrm{iem}^{+}(M)$ is path-connected, see [25].

3. Geometrical and topological preliminaries

3.1. **Conformal psc-concordance.** Let M be a closed smooth manifold with $\dim M = n-1 \geq 3$. We denote by $\mathcal{C}(M)$ the space of conformal classes of Riemannian metrics on M. There is a canonical projection map $\pi : \mathcal{R}\mathrm{iem}(M) \to \mathcal{C}(M)$ which sends a metric g to its conformal class $[g]$.

Recall that a conformal class $C \in \mathcal{C}(M)$ is called *positive* if there exists a psc-metric $g \in C$. This is equivalent to positivity of the Yamabe constant $Y_C(M)$ which is defined by the formula:

$$Y_C(M) = \inf_{g \in C} \frac{\int_M R_g d\sigma_g}{\mathrm{Vol}_g(M)^{\frac{n-3}{n-1}}}.$$

We denote the space of all positive conformal classes by $\mathcal{C}^{+}(M)$. It is known, [1, Theorem 7.1], that the projection $\pi : \mathcal{R}\mathrm{iem}(M) \to \mathcal{C}(M)$ induces weak homotopy equivalence:

$$\mathcal{R}\mathrm{iem}^{+}(M) \simeq \mathcal{C}^{+}(M).$$

In particular, the spaces $\mathcal{R}\mathrm{iem}^{+}(M)$ and $\mathcal{C}^{+}(M)$ have the same number of path components.

Now let $\bar{C}$ be a conformal class on the cylinder $M \times I$. We denote: $C_0 = \bar{C}|_{M \times \{0\}}$, $C_1 = \bar{C}|_{M \times \{1\}}$. For a given conformal class $\bar{C}$, we denote by $\bar{C}^0$ the subclass

$$\bar{C}^0 = \{ \ \bar{g} \in \bar{C} \ | \ H_{\bar{g}} \equiv 0 \ \text{ along the boundary } \} \subset \bar{C}.$$

It is easy to see that the subclass $\bar{C}^0$ is always non-empty, see [15]. Here $H_{\bar{g}}$ is the mean curvature function. In this setting, a *relative Yamabe constant*

$$Y_{\bar{C}}(M \times I, M \times \{0\} \sqcup M \times \{1\}; C_0 \sqcup C_1) = \inf_{\bar{g} \in \bar{C}^0} \frac{\int_{M \times I} R_{\bar{g}} d\sigma_{\bar{g}}}{\mathrm{Vol}_{\bar{g}}(M \times I)^{\frac{n-2}{n}}}$$

is well-defined, see [15].

Definition 3.1. Let $C_0, C_1 \in \mathcal{C}^{+}(M)$ be two positive conformal classes. We say that C_0 and C_1 are *conformally psc-concordant* if there exists a conformal class $\bar{C}$ on $M \times I$ with $C_0 = \bar{C}|_{M \times \{0\}}$, $C_1 = \bar{C}|_{M \times \{1\}}$ such that

$$Y_{\bar{C}}(M \times I, M \times \{0\} \sqcup M \times \{1\}; C_0 \sqcup C_1) > 0.$$

In that case, a conformal manifold $(M\times I, \bar{C})$ is called a *conformal psc-concordance* between the classes C_0, C_1.

It turns out the concepts of psc-concordance between psc-metrics and positive conformal classes are equivalent:

Theorem 3.2. (B. Botvinnik, K. Akutagawa, see [2, Corollary D]) *Let M be a closed manifold with $\dim M\geq 2$, and g_0, g_1 be two psc-metrics. Then the psc-metrics g_0, g_1 are psc-concordant if and only if the conformal classes $C_0=[g_0]$ and $C_1=[g_1]$ are conformally psc-concordant.*

Remark. We say that two positive conformal classes $C_0\in\mathcal{C}^+(M_0)$ and $C_1\in\mathcal{C}^+(M_1)$ are *conformally psc-bordant* if there exists a bordism W between M_0 and M_1 and a conformal class $\bar{C}$ on W such that the Yamabe constant $Y_{\bar{C}}(W, M_0\sqcup M_1; C_0\sqcup C_1)$ is positive. Then two psc-metrics g_0 and g_1 are psc-bordant if and only if the corresponding conformal classes $C_0=[g_0]$ and $C_1=[g_1]$ are conformally psc-bordant, see [2]. $\diamond$

3.2. **The space of non-negative conformal classes.** Let M be as above, a closed manifold with $\dim M = n-1\geq 3$. We denote by $\mathcal{C}^{\geq 0}(M)$ the space of non-negative conformal classes:

$$\mathcal{C}^{\geq 0}(M)=\{\ C\in\mathcal{C}(M)\mid Y_C(M)\geq 0\ \}.$$

There is a natural embedding $i:\mathcal{C}^+(M)\hookrightarrow\mathcal{C}^{\geq 0}(M)$.

Lemma 3.3. *Assume a closed manifold M does not admit a Ricci-flat metric. Then the embedding $i:\mathcal{C}^+(M)\hookrightarrow\mathcal{C}^{\geq 0}(M)$ induces an isomorphism $i_*:\pi_0\mathcal{C}^+(M)\overset{\cong}{\hookrightarrow}\pi_0\mathcal{C}^{\geq 0}(M)$.*

Proof. If $C_0, C_1\in\mathcal{C}^+(M)$ are in the same path-component of $\mathcal{C}^+(M)$, then obviously C_0, C_1 are also in the same path-component of $\mathcal{C}^{\geq 0}(M)$.

Assume that $C_0, C_1\in\mathcal{C}^+(M)$ are in the same path-component of $\mathcal{C}^{\geq 0}(M)$, and C_t, $0\leq t\leq 1$, is a continuous path in $\mathcal{C}^{\geq 0}(M)$ between C_0 and C_1. We choose a continuous path of metrics g_t with $g_t\in C_t$ lifting the path C_t. We may assume that $\mathrm{Vol}_{g_t}(M)=1$. Since $\lambda_1(L_{g_t})\geq 0$, we find a family of eigenfunctions u_t such that

$$L_{g_t}u_t=\lambda_1(L_{g_t})u_t,\qquad \int_M u_t^2 d\sigma_{g_t}=1.$$

Then the family of metrics $\tilde{g}_t=u_t^{\frac{4}{n-3}}g_t$ provides a different lift of the path C_t, and

$$R_{\tilde{g}_t}=\begin{cases} >0 & \text{if } \lambda_1(L_{g_t})>0,\\ =0 & \text{if } \lambda_1(L_{g_t})=0.\end{cases}$$

We start a family of Ricci flows $\tilde{g}_t(\tau)$ with the initial values $\tilde{g}_t(0)=\tilde{g}_t$. Since M does not admit a Ricci-flat metric, there is a short-time solution $\tilde{g}_t(\tau)$ continuously depending on the initial values. This gives a path of psc-metrics connecting $\tilde{g}_0$ and $\tilde{g}_1$ and consequently, a path of positive conformal classes between C_0 and C_1. $\square$

We do not need the following result here. However, it has an independent interest.

Theorem 3.4. *Let M be a closed manifold with $\dim M \geq 3$ which does not admit a Ricci-flat metric. Then the embedding $i : \mathcal{C}^+(M) \hookrightarrow \mathcal{C}^{\geq 0}(M)$ induces a homotopy equivalence.*

Proof. (Sketch) The argument is essentially the same as in the proof of Lemma 3.3. Indeed, instead of a path of conformal classes we should consider a compact family $\{C_\zeta\}_{\zeta\in Z}$ of conformal classes, $C_\zeta \in \mathcal{C}^{\geq 0}(M)$. Then again a family of short-time solutions of the corresponding Ricci flows provides a deformation of the family $\{C_\zeta\}_{\zeta\in Z}$ into the space $\mathcal{C}^+(M)$. This gives weak homotopy equivalence, and according to [26], this also implies an actual homotopy equivalence between the spaces $\mathcal{C}^+(M)$ and $\mathcal{C}^{\geq 0}(M)$. □

3.3. **Conformal Laplacian and minimal boundary condition.** We start by recalling necessary definitions on the conformal Laplacian on a manifold with boundary.

Let $(W, \bar{g})$ be a manifold with boundary ∂W, $\dim W = n$. We denote by $A_{\bar{g}}$ the second fundamental form along ∂W, by $H_{\bar{g}} = \operatorname{tr} A_{\bar{g}}$ the mean curvature along ∂W, and by $h_{\bar{g}} = \frac{1}{n-1} H_{\bar{g}}$ the "normalized" mean curvature. Also we denote by ∂_ν the directional derivative with respect to the outward unit normal vector field along the boundary ∂W.

Let $\tilde{g} = u^{\frac{4}{n-2}}\bar{g}$ be a conformal metric. Then we have the following standard formulas for the scalar and mean curvatures:

$$
\begin{array}{rcccl}
R_{\tilde{g}} & = & u^{-\frac{n+2}{n-2}}\left(a_n \Delta_{\bar{g}} u + R_{\bar{g}} u\right) & = & u^{-\frac{n+2}{n-2}} L_{\bar{g}} u, \qquad a_n = \frac{4(n-1)}{n-2} \\
h_{\tilde{g}} & = & \frac{2}{n-2} u^{-\frac{n}{n-2}}\left(\partial_\nu u + \frac{n-2}{2} h_{\bar{g}} u\right) & = & u^{-\frac{n}{n-2}} B_{\bar{g}} u.
\end{array}
\tag{3.1}
$$

Then the *minimal boundary problem* on $(W, \bar{g})$ is given as

$$
\left\{
\begin{array}{rcll}
L_{\bar{g}} u & = & a_n \Delta_{\bar{g}} u + R_{\bar{g}} u = \lambda_1 u & \text{on } W, \\
B_{\bar{g}} u & = & \partial_\nu u + \frac{n-2}{2} h_{\bar{g}} u = 0 & \text{on } \partial W,
\end{array}
\right.
\tag{3.2}
$$

where λ_1 is the corresponding principal eigenvalue. If u is a smooth and positive eigenfunction corresponding to the principal eigenvalue λ_1, i.e. $L_{\bar{g}} u = \lambda_1 u$, and $\tilde{g} = u^{\frac{4}{n-2}}\bar{g}$, then

$$
\left\{
\begin{array}{rcll}
R_{\tilde{g}} & = & u^{-\frac{n+2}{n-2}} L_{\bar{g}} u = \lambda_1 u^{-\frac{4}{n-2}} & \text{on } W \\
h_{\tilde{g}} & = & u^{-\frac{n}{n-2}} B_{\bar{g}} u = 0 & \text{on } \partial W.
\end{array}
\right.
\tag{3.3}
$$

3.4. **Slicing functions and pseudoisotopies.** Let M be a closed smooth manifold, as above. We take a direct product $M \times I$ and denote by $\pi_I : M \times I \to I$ the projection on the second factor.

According to Definition 2.7, a slicing function $\bar{\alpha} : M \times I \to I$ is a smooth function such that it has no critical points and it agrees with the projection $\pi_I : M \times I \to I$ near the boundary $\partial(M \times I) = M \times \{0\} \sqcup M \times \{1\}$.

We denote by $\mathcal{E}(M \times I)$ the space of slicing functions in the *Whitney topology* (known also as *weak C^∞-topology*, see [21, Chapter 1]). We denote by $\mathrm{Diff}(M \times I)$ the group of diffeomorphisms of $M \times I$ endowed also with the Whitney topology.

Then we denote by

$$\mathrm{Diff}(M \times I, M \times \{0\}) \subset \mathrm{Diff}(M \times I)$$

the subgroup of diffeomorphisms

$$\bar{\varphi} : M \times I \longrightarrow M \times I$$

such that $\bar{\varphi}|_{M\times\{0\}} = Id_{M\times\{0\}}$.

The group $\mathrm{Diff}(M \times I, M \times \{0\})$ is known as the group of *pseudo-isotopies*. There is a natural map

$$\sigma : \mathrm{Diff}(M \times I, M \times \{0\}) \longrightarrow \mathcal{E}(M \times I) \tag{3.4}$$

which sends a diffeomorphism $\bar{\varphi} : M \times I \longrightarrow M \times I$ to the function

$$\sigma(\bar{\varphi}) = \pi_I \circ \bar{\varphi} : M \times I \xrightarrow{\bar{\varphi}} M \times I \xrightarrow{\pi_I} I \ ,$$

where $\pi_I : M \times I \to I$ is as above, the projection on the second factor.

Theorem 3.5. (J. Cerf, [13]) *The map $\sigma : \mathrm{Diff}(M \times I, M \times \{0\}) \xrightarrow{\simeq} \mathcal{E}(M \times I)$ is fibration and induces homotopy equivalence.*

Remark. It is easy to see that σ is homotopy equivalence. Indeed, consider the fiber over the function $\pi_I : M \times I \to I$:

$$\sigma^{-1}(\pi_I) = \{ \ \bar{\varphi} \in \mathrm{Diff}(M \times I, M \times \{0\}) \mid \pi_I \circ \bar{\varphi} = \pi_I \ \}.$$

Then the space $\sigma^{-1}(\pi_I)$ is homeomorphic to the following space of paths in $\mathrm{Diff}(M)$:

$$\{ \ \gamma : I \to \mathrm{Diff}(M) \mid \gamma(0) = Id_M \ \}.$$

This homeomorphism is given by a map which sends a diffeomorphism $\bar{\varphi} \in \sigma^{-1}(\pi_I)$ to the path $\gamma_t : M \to M$, where $\gamma_t(x) = \bar{\varphi}(t, x)$. By definition, we have that $\gamma_0 = Id_M$. Thus the space $\sigma^{-1}(\pi_I)$ is contractible. $\diamond$

We denote by $\mathcal{F}(M \times I)$ the space of all smooth functions $M \times I \to I$ which agree with the projection $\pi_I : M \times I \to I$ near the boundary

$$\partial(M \times I) = M \times \{0\} \sqcup M \times \{1\}.$$

Clearly the space $\mathcal{F}(M\times I)$ is convex, and $\mathcal{E}(M\times I)\subset\mathcal{F}(M\times I)$. Thus, we have the isomorphism:

$$\pi_q(\mathcal{E}(M\times I))\cong\pi_{q+1}(\mathcal{F}(M\times I),\mathcal{E}(M\times I))\ . \tag{3.5}$$

The isomorphism

$$\pi_0(\mathcal{E}(M\times I))\cong\pi_1(\mathcal{F}(M\times I),\mathcal{E}(M\times I))$$

is relevant to our story and has the following geometric interpretation.

Two diffeomorphisms $\varphi_0,\varphi_1\in\mathrm{Diff}(M)$ are said to be *isotopic* if there is a smooth path $\varphi(t)$ in $\mathrm{Diff}(M)$ such that $\varphi(0)=\varphi_0$ and $\varphi(1)=\varphi_1$. The *isotopy group* $\mathcal{S}(M\times I)$ is defined to be the fiber $\sigma^{-1}(\pi_I)$, i.e.

$$\mathcal{S}(M\times I)=\{\ \bar\varphi\in\mathrm{Diff}(M\times I,M\times\{0\})\mid\pi_I\circ\bar\varphi=\pi_I\ \}.$$

Clearly $\mathcal{S}(M\times I)$ is indeed a subgroup of $\mathrm{Diff}(M\times I,M\times\{0\})$. There is a natural action

$$\mu:\mathcal{S}(M\times I)\times\mathrm{Diff}(M)\longrightarrow\mathrm{Diff}(M)$$

defined as follows. Let $\bar\psi:(x,t)\mapsto(\psi_t(x),t)$ be an isotopy, and $\varphi\in\mathrm{Diff}(M)$. Then

$$\mu(\bar\psi,\varphi)=\varphi\circ\psi_1.$$

The action μ extends to the action

$$\tilde\mu:\mathrm{Diff}(M\times I,M\times\{0\})\times\mathrm{Diff}(M)\longrightarrow\mathrm{Diff}(M)$$

which sends a pair $(\bar\varphi,\varphi)$, $\bar\varphi:M\times I\longrightarrow M\times I$ and $\varphi:M\to M$, to the diffeomorphism

$$\varphi\circ\left(\bar\varphi|_{M\times\{1\}}\right):M\to M.$$

Then two diffeomorphisms $\varphi_0,\varphi_1\in\mathrm{Diff}(M)$ are said to be *pseudo-isotopic* if there exists a pseudo-isotopy $\bar\varphi\in\mathrm{Diff}(M\times I,M\times\{0\})$ such that $\tilde\mu(\bar\varphi,\varphi_0)=\varphi_1$, i.e. $\varphi_0\circ\left(\bar\varphi|_{M\times\{1\}}\right)=\varphi_1$. By construction, two isotopic diffeomorphisms are pseudo-isotopic. The converse does not hold, in general. Clearly the obstruction is the group of path-components

$$\pi_0(\mathrm{Diff}(M\times I,M\times\{0\})\cong\pi_0(\mathcal{E}(M\times I))\cong\pi_1(\mathcal{F}(M\times I),\mathcal{E}(M\times I)).$$

The following fundamental result is proven by J. Cerf:

Theorem 3.6. (J. Cerf, [13]) *Let M be a closed simply connected manifold of dimension $\dim M\geq 5$. Then*

$$\pi_0(\mathrm{Diff}(M\times I,M\times\{0\})=\pi_1(\mathcal{F}(M\times I),\mathcal{E}(M\times I))=0.$$

In particular, any two pseudo-isotopic diffeomorphisms $\varphi_0,\varphi_1\in\mathrm{Diff}(M)$ are isotopic.

In the case when a manifold M has non-trivial fundamental group, the group

$$\pi_0(\mathrm{Diff}(M\times I,M\times\{0\})$$

is identified with a corresponding Whitehead group $\mathrm{Wh}(\pi)$ which depends on the fundamental group $\pi = \pi_1 M$. The Whitehead group $\mathrm{Wh}(\pi)$ plays a fundamental role in smooth and geometric topology, see (say, survey [23]). In particular, the obstruction group is "almost always" non-zero if the fundamental group $\pi_1 M$ contains torsison. Otherwise, it is an open question whether the Whitehead group $\mathrm{Wh}(\pi)$ is nontrivial or not for a torison-free group π (see, say, [27, Conjecture 3.4]).

Let $\mathcal{C}(M\times I)$ be the space of conformal classes on $M\times I$. The group $\mathrm{Diff}(M\times I, M\times\{0\})$ of pseudo-isotopies acts on the space of metrics $\mathcal{R}\mathrm{iem}(M\times I)$ and the space of conformal classes $\mathcal{C}(M\times I)$ by pull-back:

$$(\bar{\varphi},\bar{g})\mapsto \varphi^*\bar{g}, \qquad (\varphi,\bar{C})\mapsto \bar{\varphi}^*\bar{C}.$$

In particular, if $\bar{\varphi}\in \mathrm{Diff}(M\times I, M\times\{0\})$, and $(M\times I,\bar{g})$ is a psc-concordance (respectively, $(M\times I,\bar{C})$ is a conformal psc-concordance), then $(M\times I,\bar{\varphi}^*\bar{g})$ is also psc-concordance (respectively, $(M\times I,\bar{\varphi}^*\bar{C})$ is a conformal psc-concordance).

3.5. **Isotopy and pseudo-isotopy of diffeomorphisms versus psc-concordance.** Let us return to a conformal psc-concordance $(M\times I,\bar{C})$. We choose a metric $\bar{g}\in\bar{C}$ with zero mean curvature along the boundary. Then we choose a slicing function $\bar{\alpha}\in\mathcal{E}(M\times I)$ and construct a smooth tangent vector field

$$X_{\bar{g}}(\bar{\alpha}) = \frac{\nabla_{\bar{g}}\bar{\alpha}}{|\nabla_{\bar{g}}\bar{\alpha}|^2_{\bar{g}}}.$$

It is easy to see that

$$d\bar{\alpha}(X_{\bar{g}}(\bar{\alpha})) = \bar{g}\left\langle \nabla_{\bar{g}}\bar{\alpha}, X_{\bar{g}}(\bar{\alpha})\right\rangle = 1\ .$$

We denote by γ_x the integral curve of the vector field $X_{\bar{g}}(\bar{\alpha})$ such that $\gamma_x(0)=(x,0)$. It is easy to see that $\gamma_x(1)\in M\times\{1\}$.

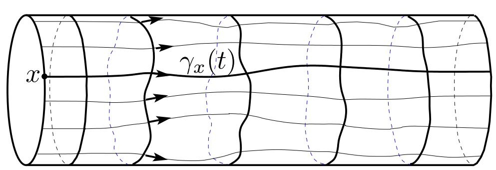

FIGURE 6. The integral curves $\gamma_x(t)$.

We obtain a diffeomorphism $\bar{\varphi} : M\times I\to M\times I$ defined by the formula

$$\bar{\varphi} : (x,t)\mapsto (\pi_M(\gamma_x(t)),\pi_I(\gamma_x(t)),$$

where $\pi_M : M\times I\to M$ and $\pi_I : M\times I\to I$ are the natural projections on the corresponding factors. Clearly $\bar{\varphi}|_{M\times\{0\}} = Id_M$, thus $\bar{\varphi}\in\mathrm{Diff}(M\times I, M\times\{0\})$ is a pseudo-isotopy.

By construction, we have that $\pi_I(\gamma_x(t)) = \bar{\alpha}(x,t)$. We introduce new coordinates:

$$(y,s) := (\pi_M(\gamma_x(t)),\pi_I(\gamma_x(t)).$$

Now we denote by $\tilde{g}(y,s)$ the metric $(\bar{\varphi}^{-1})^*\bar{g}(y,s)$. We have:

$$\begin{aligned} \tilde{g}(y,s) &= (\bar{\varphi}^{-1})^*\bar{g}(y,s) \\ &= \bar{g}|_{M_s}(y) + \frac{1}{|\nabla_{\bar{g}}\bar{\alpha}|^2_{\bar{g}}} ds^2 \\ &= \frac{1}{|\nabla_{\bar{g}}\bar{\alpha}|^2_{\bar{g}}} \left(|\nabla_{\bar{g}}\bar{\alpha}|^2_{\bar{g}} \cdot \bar{g}|_{M_s}(y) + ds^2 \right). \end{aligned}$$

We observe that $|\nabla_{\bar{g}}\bar{\alpha}|^2_{\bar{g}} \cdot \tilde{g} \in (\bar{\varphi}^{-1})^*\bar{C}$. Now we can replace the original conformal class $\bar{C}$ by the pull-back class $(\bar{\varphi}^{-1})^*\bar{C}$, and the metric $\bar{g}$ by the metric $|\nabla_{\bar{g}}\bar{\alpha}|^2_{\bar{g}} \cdot \tilde{g}$; i.e., we change the notations:

$$\begin{aligned} (y,s) &\rightsquigarrow (x,t), \\ \bar{g} &\rightsquigarrow |\nabla_{\bar{g}}\bar{\alpha}|^2_{\bar{g}} \cdot \tilde{g}, \\ |\nabla_{\bar{g}}\bar{\alpha}|^2_{\bar{g}} \cdot \bar{g}|_{M_s}(y) &\rightsquigarrow g_t, \\ \bar{C} &\rightsquigarrow (\bar{\varphi}^{-1})^*\bar{C}. \end{aligned}$$

It is easy to see that the resulting metric $\bar{g}$ has zero mean curvature along the boundary $M_0 \sqcup M_1$. Indeed, we have started with a metric which is minimal along the boundary, and the pseudo-isotopy we have applied preserves minimality since the slicing function $\bar{\alpha}$ determining the pseudoisotopy agrees with the projection $\pi_I : M \times I \to I$ near the boundary $M_0 \sqcup M_1$. We summarize the above observations:

Proposition 3.7. (K. Akutagawa) *Let $\bar{C} \in \mathcal{C}(M \times I)$ be a conformal class, and $\bar{\alpha} \in \mathcal{E}(M \times I)$ be a slicing function. Then there exists a metric $\bar{g} \in \bar{C}$ with minimal boundary condition such that*

$$\begin{cases} \bar{g} = \bar{g}|_{M_t} + dt^2 & \text{on } \ M \times I \\ \mathrm{Vol}_{g_t}(M_t) = \mathrm{Vol}_{g_0}(M_0) & \text{for all } \ t \in I \end{cases}$$

up to a pseudo-isotopy given by the slicing function $\bar{\alpha}$.

4. Cheeger-Gromov convergence for manifolds with boundary

Here we review necessary facts on Cheeger-Gromov convergence for manifolds with boundary. There are well-known results on this subject for complete manifolds or compact closed manifolds, however for manifolds with boundary, we use new approach given in the recent paper [8] by the author and O. Müller.

4.1. **Bounded geometry.** For a Riemannian metric h, we denote by Rm_h its Riemannian tensor, and by inj_h its injectivity radius. Let $(W, \bar{g}, \bar{x})$ be a pointed Riemannian manifold. In the case the manifold W has non-empty boundary ∂W, we denote by $g = \bar{g}|_{\partial W}$ the induced metric. Denote by d the distance function induced by the metric $\bar{g}$. Then for given $r > 0$ we denote by $B_r(\partial W)$ a tubular neighborhood of ∂W of radius r, i.e., $B_r(\partial W) = \{ \ x \in W \mid d(x, \partial W) < r \ \}$. In the following, we adopt the following definition of bounded geometry for manifolds with boundary, (cf. [34]):

Definition 4.1. Fix a positive integer k and a constant $\mathbf{c} > 0$. A Riemannian manifold $(W, \bar{g})$ with non-empty boundary ∂W has $(\mathbf{c}, k)$-*bounded geometry* if

(i) for the inward normal vector field ν, the normal exponential map $E : \partial W \times [0, \mathbf{c}^{-1}] \to W$, $E(y, r) := \exp_y(r\nu)$, is a diffeomorphism onto its image;
(ii) $\mathrm{inj}_g(\partial W) \ \geq \ \mathbf{c}^{-1}$;
(iii) $\mathrm{inj}_g(M \setminus B_r(\partial W)) \geq r$ for all $r \leq \mathbf{c}^{-1}$;
(iv) $|\nabla^l_{\bar{g}} \mathrm{Rm}_{\bar{g}}|_{\bar{g}} \leq c$ and $|\nabla^l_{g} \mathrm{Rm}_{\bar{g}}|_{\bar{g}} \ \leq \ \mathbf{c}$ for all $l \leq k$.

For a pointed Riemannian manifold $(W, \bar{g}, \bar{x})$ we require that $d(x, \partial W) \geq 2\mathbf{c}^{-1}$.

4.2. **Height functions.** In order to deal with convergence of manifolds with boundary, we would like to think of a Riemannian manifold with boundary as a complete Riemannian manifold equipped with extra data, namely, a *height function*. Indeed, for a manifold W with boundary, we can always attach a small collar to get a complete manifold X equipped with a height function $f : X \to (-\infty, 1)$ such that $W = f^{-1}([0, 1))$. Then a sequence $\{(W_i, \bar{g}_i, \bar{x}_i)\}$ of pointed compact manifolds with non-empty boundary gives a sequence $\{(X_i, \bar{g}_i, \bar{x}_i)\}$ (where $\bar{g}_i$ exends $\bar{g}_i$ on M_i) of complete Riemannian manifolds with additional data: height functions. In the context of psc-concordance/isotopy problem, height functions will be specified to slicing functions.

Definition 4.2. Let $(X, \bar{g}, \bar{x})$ be a pointed Riemannian manifold. A smooth function $f : X \to \mathbf{R}$ is called a $(\mathbf{c}, k)$-*height function*, where a positive integer k and $\mathbf{c} > 0$, if the following conditions are satisfied:

(i) $\delta^{\partial}(f) := \min\{ \ |\nabla_g f(x)|_g \mid x \in f^{-1}([-\varepsilon, +\varepsilon])\} \geq c^{-1}$, $f^{-1}(\{0\}) \neq \emptyset$, in particular 0 is a regular value for the function f;
(ii) $f(x) > 0$, and the distance from the base point x to the submanifold $Y^{(0)} := f^{-1}(0)$ is bounded from below by $\mathbf{c}^{-1}$ and by $\mathbf{c}$ from above.
(iii) the derivatives $|\nabla^{\ell} f| \leq \mathbf{c}$ for all $\ell = 0, 1, \ldots, k$.

A sequence $\{(W_i, \bar{g}_i, \bar{x}_i, f_i)\}$ is called of $(\mathbf{c}, k)$-bounded geometry if $\{(W_i, \bar{g}_i, \bar{x}_i)\}$ is a sequence of $(\mathbf{c}, k)$-bounded geometry and f_i are $(\mathbf{c}, k)$-height functions on W_i.

It is not difficult to see that if f is a $(\mathbf{c}, k)$-height function on a manifold of $(\mathbf{c}, k)$-bounded geometry, then $X^f = f^{-1}([0, 1))$ is a manifold with boundary of bounded geometry. It is a bit harder to see that actually also the converse is true:

Theorem 4.3. (B. Botvinnik, O. Müller, [8, Theorem 2.12]) *Let $\mathbf{c} > 0$ then there exists $\bar{\mathbf{c}} > 0$, depending on $\mathbf{c}$, such that, for any compact pointed manifold $(W, \bar{g}, \bar{x})$ (with non-empty boundary) of $(\mathbf{c}, k)$-bounded geometry, there exists a pointed isometric inclusion $\iota : (W, \bar{g}, \bar{x}) \to (X, \bar{g}, \bar{x})$ where $(X, \bar{g}, \bar{x})$ is a complete open pointed manifold of $(\bar{\mathbf{c}}, k)$-bounded geometry and $(\bar{\mathbf{c}}, k)$-height function f on X with $\iota(W) = f^{-1}([0, 1))$.*

4.3. **Gromov-Hausdorff convergence.** Here we recall some standard definitions following [4, Chapter 3]. Let Z be a metric space, and $Y \subset Z$ be a subspace. Let $B_r(Y)$ be the ball of radius r around Y in Z, where $r > 0$. In the case when $Y = \{y\}$, we use the notation $B_r(y)$ instead of $B_r(\{y\})$. Sometimes it will be important to emphasize an ambient space Z, then we use the notation $B_r^Z(y)$.

If $Z_0, Z_1 \subset Z$, then the *Hausdorff distance* $d_H(Z_0, Z_1)$ is defined as

$$d_H(Z_0, Z_1) = \inf\{ \ r > 0 \mid Z_0 \subset B_r(Z_1), \quad Z_1 \subset B_r(Z_0) \ \},$$

Let (X, d) and (X', d') are metric spaces. Then we say that a continuous map $\varphi : X \to X'$ is an ε*-isometry* if $||\varphi^* d' - d||_\infty < \varepsilon$.

Definition 4.4. Let $\{(Y_i, d_i, y_i)\}$ be a sequence of pointed proper complete metric spaces. Then the sequence $\{(Y_i, d_i, y_i)\}$ is said to *GH-converges* to a complete and proper metric pointed space $(Y_\infty, d_\infty, y_\infty)$ if one of the following equivalent conditions is satisfied:[1]

(B$'$) there are sequences $\{r_i\}$, $\{\varepsilon_i\}$ of positive real numbers, where $r_i \to \infty$, $\varepsilon_i \to 0$, and ε_i-isometries $\varphi_i : B_{r_i}^{Y_\infty}(y_\infty) \to B_{r_i}^{Y_i}(y_i)$ such that

$$B_{\varepsilon_i}(\mathrm{Im}\ \varphi_i) \supset B_{r_i}^{Y_i}(y_i) \quad \text{and} \quad d_i(\varphi_i(y_\infty), y_i) < \varepsilon_i.$$

(D$'$) there is a metric space (Z, d) and isometric embeddings $\iota_i : Y_i \to Z$, $\iota_\infty : Y_\infty \to Z$, such that
  (i) $\lim\limits_{i\to\infty} \iota_i(y_i) = \iota_\infty(y_\infty)$,
  (ii) $\lim\limits_{i\to\infty} d_H(U \cap \iota_i(Y_i), U \cap \iota_\infty(Y_\infty)) = 0$ for any open bounded set $U \subset Z$.

We use the notation $\lim\limits_{i\to\infty}^{GH}(Y_i, d_i, y_i) = (Y_\infty, d_\infty, y_\infty)$.

We need the following fact, which is a particular case of much more general results, see, for example, [4, Proposition 3.1.2, Theorem 3.1.3].

Theorem 4.5. *Let $\{(X_i, \bar{g}_i, \bar{x}_i)\}$ be a sequence of pointed complete n-dimensional Riemannian manifolds such that $\mathrm{Ric}_{g_i} \geq (n-1)\kappa$ for some $\kappa \in \mathbf{R}$ and all $i = 1, 2, \ldots$. Then there exists a pointed proper complete metric space $(Y_\infty, d_\infty, y_\infty)$ such that the sequence $\{(X_i, \bar{g}_i, \bar{x}_i)\}$ GH-subconverges to $(Y_\infty, d_\infty, y_\infty)$.*

4.4. **Smooth Cheeger-Gromov convergence.** Let $\{(X_i, \bar{g}_i, \bar{x}_i)\}$ be a sequence of pointed complete Riemannain manifolds of dimension n which GH-converges to a metric space $(Y_\infty, d_\infty, y_\infty)$ as in Definition 4.4. Assume that the metric space $(Y_\infty, d_\infty, y_\infty)$ is, in fact, a complete Riemannian manifold, and we use the notation: $(Y_\infty, d_\infty, y_\infty) = (X_\infty, \bar{g}_\infty, \bar{x}_\infty)$.

[1]We skip one more equivalent condition (A$'$), see [4, Section 3.1.2].

Definition 4.6. Assume that a sequence $\{(X_i, \bar{g}_i, \bar{x}_i)\}$ GH-converges to a complete Riemannian manifold $(X_\infty, \bar{g}_\infty, \bar{x}_\infty)$. Then the sequence $\{(X_i, \bar{g}_i, \bar{x}_i)\}$ C^k-converges to $(X_\infty, \bar{g}_\infty, \bar{x}_\infty)$ if there is an exhaustion of X_∞ by open sets U_j, i.e.,

$$U_1 \subset \cdots \subset U_j \subset \cdots \subset X_\infty, \quad X_\infty = \bigcup_j U_j,$$

and there are diffeomorphisms onto their image $\varphi_j : U_j \to X_j$ such that $\varphi_j \to Id_{X_\infty}$ pointwise, and the metrics

$$\varphi_j^* \bar{g}_j \to \bar{g}_\infty \quad C^k\text{-converging as } j \to \infty,$$

i.e., there is a point-wise convergence $\varphi_j^* \bar{g}_j \to \bar{g}_\infty$ and $\nabla^\ell \varphi_j^* \bar{g}_j \to \nabla^\ell \bar{g}_\infty$ for all $\ell = 1, \ldots, k$, where ∇ denotes the Levi-Civita connection of the metric $\bar{g}_\infty$ on X_∞.

Remark. Without loss of generalities, we will assume that a system of exhaustions $\{U_j\}$ is nothing but the systems of open balls $\{B_j(\bar{x}_\infty)\}$ of radius $j = 1, 2, \ldots$, centered at $\bar{x}_\infty \in X_\infty$.

R. Bamler provides a detailed proof (see [4, Theorem 3.2.4]) of the following result:

Theorem 4.7. (cf. R. Hamilton [19]) *Let $\{(X_i, \bar{g}_i, \bar{x}_i)\}$ be a sequence of pointed complete Riemannian manifolds of dimension n. Assume that $\mathrm{inj}_{g_i} \geq c^{-1}$ and $||\nabla^\ell \mathrm{Rm}_{\bar{g}_i}|| \leq c$ for all $\ell = 0, 1, \ldots, k$. Then the sequence $\{(X_i, \bar{g}_i, \bar{x}_i)\}$ C^k-subconverges to a pointed complete Riemannian manifold $(X_\infty, \bar{g}_\infty, \bar{x}_\infty)$ of dimension n.*

4.5. **Smooth convergence for manifolds with boundary.** Now we are ready for the convergence results we need:

Theorem 4.8. (B. Botvinnik, O. Müller, [8, Theorem 2.3]) *Let $\{(X_i, \bar{g}_i, \bar{x}_i, f_i)\}$ be a sequence of complete pointed manifolds equipped with height functions of $(\mathbf{c}, k)$-bounded geometry with $\mathbf{c} > 0$, $k \geq 4$. Then the sequence $\{(X_i, \bar{g}_i, \bar{x}_i, f_i)\}$ C^k-subconverges to $(X_\infty, \bar{g}_\infty, \bar{x}_\infty, f_\infty)$, where $(X_\infty, \bar{g}_\infty, \bar{x}_\infty)$ is a complete open manifold, and $f_\infty : X_\infty \to \mathbf{R}$ is a $(\mathbf{c}, k)$-height function.*

Corollary 4.9. *Let $\{(X_i, \bar{g}_i, \bar{x}_i, f_i)\}$ be a sequence from Theorem 4.8. Then, if we denote $W_i := X_i^{f_i}$ for $0 \leq i \leq \infty$, the sequence $\{(W_i, \bar{g}_i, \bar{x}_i)\}$ C^k-subconverges to a smooth manifold $(W_\infty, \bar{g}_\infty, \bar{x}_\infty)$ with non-empty boundary.*

4.6. **Example.** Let $(M \times I, \bar{g})$ be a psc-concordance where we choose the projection $\pi_I : M \times I \to I$ as a height function. We assume that the metric $\bar{g}$ is given as $\bar{g} = g_t + dt^2$ with respect to the coordinate system (x, t) given by the projections $\pi_I : M \times I \to I$ and $\pi_M : M \times I \to M$. Let $J_0 \subset M \times I$ be an embedded interval, such that $J_0 = (x_0, t)$, wher $x_0 \in M$ is a fixed base point, and $t \in [0, 1]$.

For each pair (t, t'), $t < t'$, we consider the Riemannian manifold $(M \times [t, t'], \bar{g}|_{M \times [t,t']})$. A linear map $\xi_{t,t'} : [0, 1] \to [t, t']$ given by the formula $\tau \mapsto (1 - \tau)t + \tau t'$ gives a diffeomorphism

$$\bar{\xi}_{t,t'} : M \times [0, 1] \to M \times [t, t'], \quad \bar{\xi}_{t,t'}(x, \tau) = (x, \xi_{t,t'}(\tau)). \tag{4.1}$$

We denote by $(\bar{M}_{t,t'}, \bar{g}_{t,t'})$ the Riemannian manifold $(M \times [0,1], \bar{g}_{t,t'})$ where the metric $\bar{g}_{t,t'}$ is given as a pull-back: $\bar{g}^{(0)}_{t,t'} = \bar{\xi}^*_{t,t'}(\bar{g}^{(0)}|_{M\times[t,t']})$. We obtain the following family of Riemannian manifolds

$$\{(\bar{M}_{t,t'}, \bar{g}_{t,t'}, J_0, \bar{x}_{t,t'})\}_{0\leq t\leq t'\leq 1}, \tag{4.2}$$

where the base point $\bar{x}_{t,t'} \in \bar{M}_{t,t'}$ is the mid-point of the embedded interval $J_0 \subset \bar{M}_{t,t'}$.

Now we fix $t^* \in (0,1)$ and choose two sequences $t_j \nearrow t^*$, $t'_j \searrow t^*$, where $t_j < t^* < t'_j$. Consider the sequence of Riemannian manifolds

$$\{(\bar{M}_{t_j,t'_j}, \bar{g}_{t_j,t'_j}, \bar{x}_{t_j,t'_j})\}. \tag{4.3}$$

The following lemma is straightforward.

Lemma 4.10. *The sequence* (4.3) *is C^k-converging to a cylindrical manifold, i.e.,*

$$\lim_{j\to\infty} (\bar{M}_{t_j,t'_j}, \bar{g}_{t_j,t'_j}, \bar{x}_{t_j,t'_j}) = (M \times I, g_{t^*} + dt^2, \bar{x}^*), \tag{4.4}$$

where g_{t^*} is the restriction of $\bar{g}$ to the slice M_{t^*}.

4.7. **Λ-function associated to a conformal psc-concordance.** Consider a conformal psc-concordance $(M \times I, \bar{C})$, and choose a metric $\bar{g} \in C$ with minimal boundary condition, and we fix a slicing function $\bar{\alpha} : M \times I \to I$. Recall that in Section 2.6, we have constructed a family of Riemannian manifolds $\{(\bar{M}_{t,t'}, \bar{g}_{t,t'}, \bar{x}_{t,t'})\}_{0\leq t<t'\leq 1}$ parametrized by the half-closed triangle $T = \{ (s,t) \mid 0 \leq s < t \leq 1 \}$. Here we used the horizontal stretching (2.3) from $\bar{M}^*_{t,t'} = \bar{\alpha}^{-1}([t,t'])$ to $\bar{M}_{t,t'} \cong M \times [0,1]$, see Fig. 4. Then we considered the minimal boundary problem (2.4) on each manifold $(\bar{M}_{t,t'}, \bar{g}_{t,t'})$ which gave us the function

$$\Lambda_{(M\times I,\bar{g},\bar{\alpha})} : T \to \mathbf{R}, \quad \Lambda_{(M\times I,\bar{g},\bar{\alpha})} : (t,t') \mapsto \lambda_1(L_{\bar{g}_{t,t'}}), \tag{4.5}$$

by evaluating the principal eigenvalues of the corresponding minimal boundary problem (2.4).

Definition 4.11. The function $\Lambda_{(M\times I,\bar{g},\bar{\alpha})}$ is called Λ*-function associated to the triple* $(M \times I, \bar{g}, \bar{\alpha})$. We say that the triple $(M \times I, \bar{g}, \bar{\alpha})$ is *non-negative* if its Λ-function is non-negative.

We denote by $\bar{T}$ the closed triangle $\bar{T} = \{ (s,t) \mid 0 \leq s \leq t \leq 1 \}$. The following proposition is a consequence of Lemma 4.10.

Proposition 4.12. *The function $\Lambda_{(M\times I,\bar{g},\bar{\alpha})}$ satisfies the following properties:*

(a) *the function $\Lambda_{(M\times I,\bar{g},\bar{\alpha})}$ is continuous on T and there is a unique continious extension $\bar{\Lambda}_{(M\times I,\bar{g},\bar{\alpha})} : \bar{T} \to \mathbf{R}$;*
(b) *the normalized eigenfunctions $u_{t,t'}$ of $L_{\bar{g}_{t,t'}}$ corresponding to a principal eigenvalue with minimal boundary condition on $\bar{M}_{t,t'}$ continuously depend on $(t,t') \in \bar{T}$.*

Clearly **(a)** and **(b)** imply that the limit $\bar{\Lambda}_{(M\times I,\bar{g},\bar{\alpha})}(s) = \lim_{t\to t'} \Lambda_{t,t'}$ exists on all $t' \in I$.

5. Proof of Theorem 2.8

5.1. **Almost conformal Laplacian.** Let M be a closed manifold as above with $\dim M = n-1 \geq 3$. For a given Riemannian metric g on M, we consider the elliptic operator

$$\mathcal{L}_g = a_n \Delta_g + R_g, \quad \text{where} \quad a_n = \tfrac{4(n-1)}{n-2}.$$

We call $\mathcal{L}_g$ *almost conformal Laplacian*, see [3, Section 2]: it is slightly different from the conformal Laplacian $L_g = a_{n-1}\Delta_g + R_g$. Since

$$a_{n-1} - a_n = \tfrac{4}{(n-3)(n-2)}$$

the difference $\mathcal{L}_g - L_g$ is a positive operator. This implies the following fact, see [3, Lemma 2.10].

Lemma 5.1. *Let $\lambda_1(\mathcal{L}_g)$ be the principal eigenvalue of the operator $\mathcal{L}_g$.*

(i) *If $\lambda_1(\mathcal{L}_g) > 0$, then $\lambda_1(L_g) > 0$. Thus $\lambda_1(\mathcal{L}_g) > 0$ implies that $[g] \in \mathcal{C}^+(M)$.*

(ii) *If $\lambda_1(\mathcal{L}_g) = 0$. Then either $R_g \equiv 0$, or there exits a metric $\tilde{g} \in [g]$ such that $R_{\tilde{g}} > 0$. Thus $\lambda_1(\mathcal{L}_g) = 0$ implies that $[g] \in \mathcal{C}^{\geq 0}(M)$.*

5.2. **Proof of Theorem 2.8 in a special case.** Let $(M \times I, \bar{g})$ be a psc-concordance. In addition to the assumption of Theorem 2.8, we assume that

(1) the slicing function $\bar{\alpha}$ coincides with the projection $\pi_I : M \times I \to I$;

(2) the metric $\bar{g}$ is given as $\bar{g} = g_t + dt^2$.

We choose $t^* \in I$ and a nested sequence of intervals $\{[t_k, t'_k]\}$ such that $t_k \to t^*$ and $t'_k \to t^*$ as $k \to \infty$. Then we have the manifolds $(\bar{M}^*_{t_k,t'_k}, \bar{g}^*_{t_k,t'_k})$, where

$$\bar{M}^*_{t_k,t'_k} = \bar{\alpha}^{-1}([t_k, t'_k]), \quad \bar{g}^*_{t_k,t'_k} = \bar{g}|_{\bar{M}^*_{t_k,t'_k}}.$$

Then we use a stretching as above to construct the pointed manifolds $(\bar{M}_{t_k,t'_k}, \bar{g}_{t_k,t'_k}, \bar{x}_{t_k,t'_k})$, and denote by $\lambda_1(L_{\bar{g}_{t_k,t'_k}})$ the principal eigenevalue of the minimal boundary problem on the manifold $(\bar{M}_{t_k,t'_k}, \bar{g}_{t_k,t'_k})$. We obtain a sequence of Riemannian manifolds $(\bar{M}_{t_k,t'_k}, \bar{g}_{t_k,t'_k}, \bar{x}_{t_k,t'_k})$ as in (4.3). Recall that $\bar{g}_{t,t'} = \bar{\xi}^*_{t,t'}(\bar{g}^*_{t,t'})$ is the metric on $\bar{M}_{t,t'}$. Then by Lemma 4.10, the sequence $(\bar{M}_{t_k,t'_k}, \bar{g}_{t_k,t'_k}, \bar{x}_{t_k,t'_k})$ converges to the cylindrical manifold $(M \times I, g_{t^*} + dt^2)$ and by construction,

$$\lambda_1(L_{\bar{g}_{t_k,t'_k}}) \to \lambda_1(L_{g_{t^*}+dt^2}),$$

where $\lambda_1(L_{g_{t^*}+dt^2})$ is the principal eigenvalue of the minimal boundary problem on the cylindrical manifold $(M \times I, g_{s^o} + dt^2)$. It is easy to see that

$$\lambda_1(L_{g_{t^*}+dt^2}) = \lambda_1(\mathcal{L}_{g_{t^*}}),$$

where $\mathcal{L}_{g_t^*}$ is the almost conformal Laplacian on (M_{t^*}, g_{t^*}). According to Proposition 4.12, the limit

$$\lim_{k\to\infty} \lambda_1(L_{\bar{g}_{t_k,t'_k}})$$

exists, and by the assumption, $\lambda_1(L_{\bar{g}_{t_k,t'_k}}) \geq 0$. Thus $\lambda_1(\mathcal{L}_{g_t^*}) \geq 0$ for each $t^* \in [0,1]$.

According to Lemma 5.1, the conformal classes $C_t = [g_t]$ are all non-negative. Then the Ricci flow argument given in Section 2.7 completes the proof.

5.3. **The general case.** Let $(M\times I, \bar{C})$ be a conformal psc-concordance between positive conformal classes C_0 and C_1, and $\bar{g} \in \bar{C}$ be a metric with zero mean curvature along the boundary. Assume that $\Lambda_{(M\times I,\bar{g},\bar{\alpha})} \geq 0$ for a given slicing function $\bar{\alpha} : M \times I \to I$.

Then, according to section 3.5, namely, Proposition 3.7, we may assume that up to pseudo-isotopy, our metric $\bar{g}$ is given as $\bar{g} = g_t + dt^2$, and the slicing function $\bar{\alpha}$ coincides with the projection $\pi_I : M \times I \to I$. This is exactly the "special case" that we just have proved above. This concludes our proof of Theorem 2.8. □

6. Preparations for the proof of Theorem 2.9

6.1. **First steps.** We prove Theorem 2.9 by contradiction. Namely, assuming the result of Theorem 2.9 fails, we choose a counterexample, a compact manifold M and two conformaly psc-concordant conformal classes C_0 and C_1 such that the conclusion given in Theorem 2.9 fails. Then we choose some conformal psc-concordance $(M \times I, \bar{C})$ between C_0 and C_1 and a slicing function $\bar{\alpha}$.

Next, we use a pseudoisotopy and Proposition 3.7 to find a psc-concordance $(M \times I, \bar{g})$ where the metric $\bar{g}$ is *equidistant* with respect to the standard projection $M\times I \to I$, i.e., $\bar{g} = g_t + dt^2$. We say that the metric $\bar{g} = g_t + dt^2$ is *originated from the C-counterexample* (M, C_0, C_1). We choose a base point $x_0 \in M$ and denote by J_0 the interval $J_0 = \{x_0\} \times I$. Let $g^{(n-1)}_{\text{torp}}(\varepsilon)$ be the torpedo metric on D^{n-1}, see [37, Section 1.3].

Definition 6.1. Let $\varepsilon > 0$. We say that a metric $\bar{g}$ on $M \times I$ is *ε-standard along the interval J_0* if $\bar{g}$ restricts to the metric $g^{(n-1)}_{\text{torp}}(\varepsilon) + dt^2$ on the product $D^{n-1} \times I \subset M \times I$. We use the notation $(M \times I, \bar{g}, J_0)$ to emphasize that the metric $\bar{g}$ is ε-standard along J_0 (and we suppress ε from the notations).

Lemma 6.2. *Let $\bar{g} = g_t + dt^2$ be the metric originated from a C-counterexample (M, C_0, C_1), and $x_0 \in M$ be a base point as above. Then there exists a C-counterexample (M, C'_0, C'_1) and a path $\bar{g}' = g'_t + dt^2$ originated from some C-counterexample, such that the metric $\bar{g}'$ is ε-standard along J_0 for some $\varepsilon > 0$.*

Proof. By assumption, the metric $\bar{g} = g_t + dt^2$ could be considered as a conformal psc-concordance, i.e. when $C_0 = [g_0]$, $C_1 = [g_1]$ and $\bar{C} = [\bar{g}]$. Then for each t, there exists a deformation of the

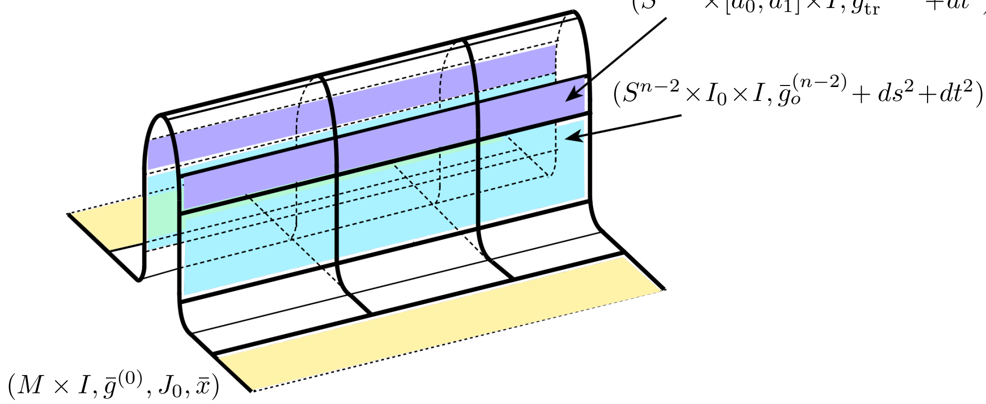

FIGURE 7. A concordance $(M\times I, \bar{g}^{(0)}, J_0, \bar{x})$ near the interval J_0.

metrics g_t to a metric $g_t^{(1)}$ which restricts to a torpedo metric $g^{(n-1)}_{\mathrm{torp}}(\varepsilon)$ near the base point x_t (for some ε). The deformation could be chosen to be smooth dependent on t, and we may assume that ε is the same for all t by compactness. This deformation changes conformal classes C_i to C_i', $i=0,1$, however the metrics g_i and g_i' are still in the same path components of the space $\mathcal{R}\mathrm{iem}^{\lambda_1\geq 0}(M)$.[2] Then this deformation creates a conformal psc-concordance $(M, g_t'+dt^2)$, and the triple (M, C_0', C_1') has to be a C-counterexample since (M, C_0, C_1) is. $\square$

We choose small $\varepsilon_0>0$ and use Lemma 6.2 to adjust the metric $\bar{g}$ to make it ε_0-*standard* near the interval J_0. The resulting metric is denoted by $\bar{g}^{(0)} = g_t^{(0)}+dt^2$, where $g_t^{(0)}$ is a metric on the slice $M_t = M\times\{t\}$. Now we fix such a manifold $(M\times I, \bar{g}^{(0)}, J_0, \bar{x})$, where the base point $\bar{x}\in M\times I$ is the mid-point of the interval $J_0\subset M\times I$.

For the future use, we recall that the metric $g^{(n-1)}_{\mathrm{torp}}(\varepsilon)+dt^2$ on the product $D^{n-1}\times I$ is decomposed as follows, see Fig. 7, where $I_0=[0,a_0]$:

$$
\begin{aligned}
(D^{n-1}\times I, g^{(n-1)}_{\mathrm{torp}}(\varepsilon)+dt^2) &= (S^{n-2}\times[0,a_0]\times I, \bar{g}^{(n-2)}_{o}+ds^2+dt^2)\\
&\cup\ (S^{n-2}\times[a_0,a_1]\times I, \bar{g}^{(n-2)}_{\mathrm{tr}}+dt^2)\cup(S^{n-1}_{+}(\varepsilon)\times I, g^{(n-1)}_{o}+dt^2).
\end{aligned}
\tag{6.1}
$$

for some fixed $0<a_0<a_1<1$.

6.2. A family of manifolds $\boldsymbol{\mathcal{W}}^{(0)}$.

We define a family of Riemannian manifolds $\boldsymbol{\mathcal{W}}^{(0)}$ determined by the initial psc-concordance $(M\times I, \bar{g}^{(0)}, J_0, \bar{x})$.

First, for each pair (t,t'), $t<t'$, we consider the Riemannian manifold $(M\times[t,t'], \bar{g}^{(0)}|_{M\times[t,t']})$. A linear map $\xi_{t,t'}:[0,1]\to[t,t']$ given by the formula $\tau\mapsto(1-\tau)t+\tau t'$ gives a diffeomorphism

$$
\bar{\xi}_{t,t'}: M\times[0,1]\to M\times[t,t'],\quad \bar{\xi}_{t,t'}(x,\tau)=(x,\xi_{t,t'}(\tau)). \tag{6.2}
$$

[2]Here $\mathcal{R}\mathrm{iem}^{\lambda_1\geq 0}(M)$ is the subspace $\{\, g \mid \lambda_1(L_g)\geq 0\,\}$ of $\mathcal{R}\mathrm{iem}(M)$

We denote by $(\bar{M}_{t,t'}, \bar{g}^{(0)}_{t,t'})$ the Riemannian manifold $(M\times[0,1], \bar{g}^{(0)}_{t,t'})$ where the metric $\bar{g}^{(0)}_{t,t'}$ is given as a pull-back: $\bar{g}^{(0)}_{t,t'} = \bar{\xi}^{*}_{t,t'}(\bar{g}^{(0)}|_{M\times[t,t']})$. By construction, the ε_0-standard metric along the interval J_0 is invariant under the diffeomorphisms $\bar{\xi}_{t,t'}$, so we obtain the following family of Riemannian manifolds generated by the counterexample $(M\times I, \bar{g}^{(0)}, J_0, \bar{x})$:

$$\boldsymbol{\mathcal{W}}^{(0)} := \boldsymbol{\mathcal{W}}(M\times I, \bar{g}^{(0)}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})\}_{0\leq t\leq t'\leq 1}, \tag{6.3}$$

equipped with ε_0-standard metric along J_0. Here we always choose a canonical base point $\bar{x}_{t,t'}\in \bar{M}_{t,t'}$, the mid-point of the embedded interval $J_0\subset \bar{M}_{t,t'}$. We consider the above construction as a function:

$$(M\times I, \bar{g}, J_0, \bar{x}) \mapsto \boldsymbol{\mathcal{W}}(M\times I, \bar{g}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \bar{g}_{t,t'}, J_0, \bar{x}_{t,t'})\}_{0\leq t\leq t'\leq 1},$$

which is defined for any equidistant (and ε_0-standard along J_0) metric $\bar{g} = g_t + dt^2$.

Remark. It follows from the construction that the metric $\bar{g}^{(0)}_{t,t'}$ restricted to the boundary $\partial\bar{M}_{t,t'} = M_t \sqcup M_{t'}$ coincides with the original metrics $g_t\sqcup g_{t'}$. We emphasize that the family $\boldsymbol{\mathcal{W}}(M\times I, \bar{g}, J_0, \bar{x})$ is parametrized by a closed triangle $\bar{T}$ from Proposition 4.12. In particular if $t=t'$, the manifold $(\bar{M}_{t,t}, \bar{g}^{(0)}_{t,t}, J_0, \bar{x}_{t,t'})$ is isometric to the cylinder $(M\times I, g_{t'}+dt^2, J_0, \bar{x}_{t,t'})$.

6.3. **Again: manifolds with bounded geometry.** We observe the following property of the family $\boldsymbol{\mathcal{W}}(M\times I, \bar{g}^{(0)}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})\}$:

Lemma 6.3. *Let $k\geq 8+2n$. There exists a constant $\mathbf{c}^{(0)}>0$ such that each manifold $(\bar{M}_{t,t'}, \bar{g}_{t,t'}, \bar{x}_{t,t'})$ in the family $\boldsymbol{\mathcal{W}}(M\times I, \bar{g}^{(0)}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})\}$ has $(\mathbf{c}^{(0)},k)$-bounded geometry.*

Proof. Indeed, each manifold $(\bar{M}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})$ has $(\mathbf{c}_{t,t'},k)$-bounded geometry in the sense of Definition 4.1 for some constant $\mathbf{c}_{t,t'}>0$. The family $\boldsymbol{\mathcal{W}}^{(0)} = \{(\bar{M}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})\}$ is paramentrized by the closed triangle $\bar{T}$. Thus the constant $\mathbf{c}^{(0)}>0$ exists by compactness. ☐

6.4. **Kobayashi tubes.** To proceed further, we need a particular manifold

$$(S^{n-1}\times I, \bar{\hbar}^{(\ell)}), \quad \bar{\hbar}^{(\ell)} = \hbar^{(\ell)} + dt^2,$$

where $\hbar^{(\ell)}$ is the Kobayashi metric on S^{n-1} defined below.

Proposition 6.4. (O. Kobayashi [22]) *Let ℓ be any positive integer and $n-1\geq 3$. Then there exists a psc-metric $\hbar^{(\ell)}$ on the sphere S^{n-1} such that*

(a) $R_{\hbar^{(\ell)}} > \ell$, $\lambda_1(L_{\hbar^{(\ell)}}) > \ell$,

(b) $\mathrm{Vol}_{\hbar^{(\ell)}}(S^{n-1}) = 1$.

Definition 6.5. We fix $\ell>0$ for the rest of the paper, and denote $\hbar := \hbar^{(\ell)}$. We call a metric $\hbar$ satisfying the conditions (a) and (b) above, a *Kobayashi metric*, and the Riemannain manifold $(S^{n-1},\hbar)$ a *Kobayashi sphere*.

Remark. The metric $\hbar$ could be constructed by taking a connected sum of standard spheres; such metric is otherwise known as a "dumbbell metric". $\diamond$

We choose $\varepsilon > 0$ and choose the north and south poles $y', y'' \in S^{n-1}$ together with disks D', D'' of radius $\varepsilon > 0$ centered at those points.

Lemma 6.6. *There exists $\varepsilon > 0$ and a Kobayashi metric $\hbar$ satisfying the conditions* (a) *and* (b) *from Proposition 6.4, such that $\hbar|_D = g_{\mathrm{torp}}^{(n-1)}(\varepsilon)$, where $D = D', D''$, and $g_{\mathrm{torp}}^{(n-1)}(\varepsilon)$ is a torpedo metric of radius ε centered at the points y', y'' respectively.*

Remark. Below we always assume that the Kobayashi sphere $(S^{n-1}, \hbar)$ comes together with two fixed points y_0, y_1, and a constant $\varepsilon > 0$ as above, and we assume $\varepsilon_0 = \varepsilon$. Moreover, a choice of fixed points allows us to take connected sum of the Kobayashi spheres in a canonical way. $\diamond$

Definition 6.7. Let $(S^{n-1}, \hbar)$ be a sphere equipped with a metric $\hbar$, and $y', y'' \in S^{n-1}$ be the north and south poles of the sphere, and

$$J_0' = \{y'\} \times I \subset S^{n-1} \times I, \quad J_0'' = \{y''\} \times I \subset S^{n-1} \times I.$$

We define the metric $\bar{\hbar} = \hbar + dt^2$ and two base points $\bar{y}' = \{y'\} \times \{1/2\}$ and $\bar{y}'' = \{y''\} \times \{1/2\}$ on $S^{n-1} \times I$. Then we say that the manifold

$$(S^{n-1} \times I, \bar{\hbar}, J_0', \bar{y}', J_0'', \bar{y}'')$$

equipped with the above data is *Kobayashi tube* if it satisfies the following conditions:

(1) the metric $\hbar$ is a Kobayashi metric;

(2) the metric $\bar{\hbar}$ is ε_0-standard along the intervals J_0' and J_0'', and the base points $\bar{y}'$ and $\bar{y}''$ are the mid-points of the intervals J_0' and J_0'' respectively.

We recall the following facts [5, Lemmas 6.5, 6.6]:

Lemma 6.8. *Let $\mathcal{L}_\hbar = a_n \Delta_\hbar + R_\hbar$ be the almost conformal Laplacian, and $c_0 > 0$ be a given constant, and $\hbar = \hbar^{(\ell)}$. Then there exists $\ell \geq 2$ such that $\lambda_1(\mathcal{L}_\hbar) > c_0$, where $\lambda_1(\mathcal{L}_\hbar)$ is the principal eigenvalue of $\mathcal{L}_\hbar$.*

Lemma 6.9. *Let $(S^{n-1} \times I, \bar{\hbar}, J_0', \bar{y}', J_0'', \bar{y}'')$ be a Kobayshi tube, and $c_2 > 0$ be a given constant, $\hbar = \hbar^{(\ell)}$. Then for any ℓ such that $\frac{3}{4} a_n \lambda_1(\mathcal{L}_\hbar) > c_2^2$, there exists a metric $\tilde{\hbar} \in [\bar{\hbar}]$ such that*

(1) $R_{\tilde{\hbar}} \equiv 0$,

(2) $\mu_1(L_{\tilde{\hbar}}) \geq c_2$.

Remark. We assume that we already fixed ℓ and c_2 such that Lemma 6.9 holds. $\diamond$

6.5. **Adjustment of the family $\boldsymbol{\mathcal{W}}^{(0)}$.** Now we would like to adjust the family $\boldsymbol{\mathcal{W}}^{(0)} = \{(\bar{M}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})\}$ by relaxing the bounds given by the constant $\mathbf{c}^{(0)}$ as follows. We consider again the initial concordance $(M\times I, \bar{g}^{(0)}, J_0, \bar{x})$, where, in particular, the metric $\bar{g}^{(0)}$ is ε_0-standard along the interval $J_0 = \{x_0\}\times I$. Since the torpedo metrics are isometric near the corresponding base points, we can glue together $(M\times I, \bar{g}^{(0)}, \bar{x})$, and the Kobayashi tube $(S^{n-1}\times I, \bar{h}, J_0', \bar{y}', J_0'', \bar{y}'')$ along the isometric strips

$$D^{n-1}\times J_0 \subset M\times I, \quad D^{n-1}\times J_0' \subset S^{n-1}\times I \tag{6.4}$$

as follows. Recall that the strips $D^{n-1}\times J_0$ and $D^{n-1}\times J_0'$ are decomposed as it was given in (6.1), see also Fig. 7:

$$\begin{aligned}(D^{n-1}\times J_0, g^{(n-1)}_{\mathrm{torp}}(\varepsilon)+dt^2) &= (S^{n-2}\times[0,a_0]\times J_0, \bar{g}^{(n-2)}_o + ds^2+dt^2)\\ &\cup (S^{n-2}\times[a_0,a_1]\times J_0, \bar{g}^{(n-2)}_{\mathrm{tr}}+dt^2)\cup(S^{n-1}_+(\varepsilon)\times J_0, g^{(n-1)}_o+dt^2);\end{aligned}$$

and

$$\begin{aligned}(D^{n-1}\times J_0', g^{(n-1)}_{\mathrm{torp}}(\varepsilon)+dt^2) &= (S^{n-2}\times[0,a_0]\times J_0', \bar{g}^{(n-2)}_o + ds^2+dt^2)\\ &\cup (S^{n-2}\times[a_0,a_1]\times J_0', \bar{g}^{(n-2)}_{\mathrm{tr}}+dt^2)\cup(S^{n-1}_+(\varepsilon)\times J_0', g^{(n-1)}_o+dt^2).\end{aligned}$$

We define two intermediate manifolds:

$$\begin{aligned}(N^{(0)}, \check{g}^{(0)}) &:= (M\times I, \bar{g}^{(0)}, J_0, \bar{x})\setminus(S^{n-1}_+(\varepsilon)\times J_0, g^{(n-1)}_o+dt^2),\\ (N^{(1)}, \check{g}^{(1)}) &:= (S^{n-1}\times I, \bar{h}, J_0', \bar{y}', J_0'', \bar{y}'')\setminus(S^{n-1}_+(\varepsilon)\times J_0', g^{(n-1)}_o+dt^2).\end{aligned}$$

Then we glue $(N^{(0)}, \check{g}^{(0)})$ and $(N^{(1)}, \check{g}^{(1)})$ together by identifying the manifolds

$$\begin{aligned}&(S^{n-2}\times[0,a_0]\times J_0, \bar{g}^{(n-2)}_o + ds^2+dt^2) \quad\subset\quad (N^{(0)}, \check{g}^{(0)}) \quad \text{and}\\ &(S^{n-2}\times[0,a_0]\times J_0', \bar{g}^{(n-2)}_o + ds^2+dt^2) \quad\subset\quad (N^{(1)}, \check{g}^{(1)})\end{aligned}$$

via the formula $(x,s,t)\mapsto(x, a_0-s, t)$, where $x\in S^{n-2}$, $s\in[0,a_0]$, $t\in J_0\cong J_0'$.

We denote by $(M\times I, \bar{g}^{(1)}, J_0, \bar{x})$ the resulting manifold, where we rename the interval J_0'' by J_0, and new base point $\bar{x}\in M\times I$ coincides with the base point $\bar{y}''\in J_0''$. The manifold $(M\times I, \bar{g}^{(1)}, J_0)$ gives new family

$$\boldsymbol{\mathcal{W}}^{(1)} := \boldsymbol{\mathcal{W}}(M\times I, \bar{g}^{(1)}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \bar{g}^{(1)}_{t,t'}, J_0, \bar{x}_{t,t'})\}_{0\le t\le t'\le 1}, \tag{6.5}$$

equipped with ε_0-standard metric along J_0. We notice that alternatively the family $\boldsymbol{\mathcal{W}}^{(1)}$ could be constructed by attaching the Kobayashi tube to each manifold $(\bar{M}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})$.

Now we find new constant $\mathbf{c}^{(1)}$ such that all manifolds $(\bar{M}_{t,t'}, \bar{g}^{(1)}_{t,t'}, J_0, \bar{x}_{t,t'})$ in the family (6.5) have $(\mathbf{c}^{(1)}, k)$-bounded geometry. Let

$$\boldsymbol{\mathcal{W}}^{(q)} := \boldsymbol{\mathcal{W}}(M \times I, \bar{g}^{(k)}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \bar{g}^{(k)}_{t,t'}, J_0, \bar{x}_{t,t'})\}_{0\le t\le t'\le 1}$$

be a family which we obtain by repeating this gluing procedure q times, i.e., we keep attaching the Kobayashi tube to each manifold $(\bar{M}_{t,t'}, \bar{g}^{(1)}_{t,t'}, J_0, \bar{x}_{t,t'})$. Clearly, we do not change the geometrical bounds by attaching new Kobayashi tubes. Thus the following property of the family $\boldsymbol{\mathcal{W}}^{(q)}$ holds by construction:

Proposition 6.10. *The family of manifolds* $\boldsymbol{\mathcal{W}}^{(q)} = \boldsymbol{\mathcal{W}}(M \times I, \bar{g}^{(q)}, J_0) = \{(\bar{M}_{t,t'}, \bar{g}^{(q)}_{t,t'}, J_0, \bar{x}_{t,t'})\}$ *has the same* $(\mathbf{c}^{(1)}, k)$*-bounded geometry for all* $q = 1, 2, \ldots$.

7. Scalar-flat and minimal-boundary satellites

7.1. **Scalar-flat boundary problem.** Let $(W, \bar{g})$ be a compact Riemannian manifold with non-empty boundary $(\partial W, g)$, $\dim W = n$. We denote by $L_{\bar{g}}$ the conformal Laplacian on W, and by $h_{\bar{g}}$ the normalized mean curvature function along ∂W. We consider the following pair of operators:

$$\begin{cases} L_{\bar{g}} &= a_n \Delta_{\bar{g}} + R_{\bar{g}} \quad \text{on } W, \\ \\ B_{\bar{g}} &= \partial_\nu + b_n h_{\bar{g}} \quad \text{on } \partial W. \end{cases}$$

Here $a_n = \frac{4(n-1)}{n-2}$ and $b_n = \frac{n-2}{2}$, as usual. Here is a relevant Rayleigh quotient, where we take the infimum:

$$\mu_1 = \inf_{f \in C^\infty_+} \frac{\int_W (a_n |\nabla_{\bar{g}} f|^2 + R_{\bar{g}} f^2) d\sigma_{\bar{g}} + 2(n-1) \int_{\partial W} h_{\bar{g}} f^2 d\sigma_g}{\int_{\partial W} f^2 d\sigma_g}. \tag{7.1}$$

According to the standard elliptic theory, we obtain an elliptic boundary problem which will be denoted by $(L_{\bar{g}}, B_{\bar{g}})^\flat$, and the infimum μ_1 is the *principal eigenvalue of the boundary problem* $(L_{\bar{g}}, B_{\bar{g}})^\flat$. In paricular, there exists a smooth positive principal eigenfunction $\tilde{v}$ minimizing the functional (7.1) which satisfies the corresponding Euler-Lagrange equations

$$\begin{cases} L_{\bar{g}} \tilde{v} &= a_n \Delta_{\bar{g}} \tilde{v} + R_{\bar{g}} \tilde{v} \equiv 0 \quad \text{on } W, \\ \\ B_{\bar{g}} \tilde{v} &= \partial_\nu \tilde{v} + b_n h_{\bar{g}} \tilde{v} = \mu_1 \tilde{v} \quad \text{on } \partial W. \end{cases} \tag{7.2}$$

The eigenfunction $\tilde{v}$ is usually normalized as $\int_{\partial W} \tilde{v}^2 d\sigma_g = 1$, however, we will use different normalization below, see (7.4). We notice that for the conformal metric $\tilde{g} = \tilde{v}^{\frac{4}{n-2}} \bar{g}$ we have:

$$\begin{cases} R_{\tilde{g}} &= \tilde{v}^{-\frac{n+2}{n-2}} L_{\bar{g}} \tilde{v} \equiv 0 \quad \text{on } W, \\ \\ h_{\tilde{g}} &= \tilde{v}^{-\frac{n}{n-2}} B_{\bar{g}} \tilde{v} = \mu_1 \tilde{v}^{-\frac{2}{n-2}} \quad \text{on } \partial W. \end{cases}$$

Thus the conformal metric $\tilde{g}$ is scalar-flat and its mean curvature has a definite sign, the same as the principal eigenvalue μ_1.

Definition 7.1. We refer to the boundary problem $(L_{\bar{g}}, B_{\bar{g}})^\flat$, and the Euler-Lagrange equations (7.6) as the *scalar-flat boundary problem on* $(W,\bar{g})$, and we call the Riemannian manifold $(W,\tilde{g})$ the *scalar-flat satellite of* $(W,\bar{g})$.

7.2. **A family of scalar-flat satellites $\boldsymbol{\mathcal{W}}_\flat$.** Here we consider again a manifold $(M\times I, \bar{g}, J_0, \bar{x})$, where the metric $\bar{g} = g_t + dt^2$ is ε_0-standard along the interval J_0. Consider the family of manifolds as above $\boldsymbol{\mathcal{W}}(M\times I, \bar{g}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \bar{g}_{t,t'}, J_0, \bar{x}_{t,t'})\}$. Then for each pair (t,t'), $t\leq t'$, we consider the scalar-flat elliptic boundary problem $(L_{\bar{g}_{t,t'}}, B_{\bar{g}_{t,t'}})^\flat$ on $(\bar{M}_{t,t'}, \bar{g}_{t,t'}, J_0, \bar{x}_{t,t'})$, as above. Then we find the eigenfunction $\tilde{v}$ solving the scalar-flat boundary problem as in (7.2). We obtain

$$
\left\{
\begin{array}{lcll}
L_{\bar{g}_{t,t'}}\tilde{v} & = & a_n\Delta_{\bar{g}_{t,t'}}\tilde{v} + R_{\bar{g}_{t,t'}}\tilde{v} = 0 & \text{on } \bar{M}_{t,t'}, \\[2ex]
B_{\bar{g}_{t,t'}}\tilde{v} & = & \partial_\nu \tilde{v} + b_n h_{\bar{g}_{t,t'}} v = \mu_1(t,t')\tilde{v} & \text{on } \partial\bar{M}_{t,t'} = M_t \sqcup -M_{t'}.
\end{array}
\right. \tag{7.3}
$$

Here the eigenfunction $\tilde{v}$ depends on (t,t'), however, we suppress this dependence in the notations. We let $\tilde{g}_{t,t'} = \tilde{v}^{\frac{4}{n-2}}\bar{g}_{t,t'}$ be a corresponding conformal metric, then

$$
\left\{
\begin{array}{lcll}
R_{\tilde{g}_{t,t'}} & = & \tilde{v}^{-\frac{n+2}{n-2}} L_{\bar{g}_{t,t'}}\tilde{v} \equiv 0 & \text{on } \bar{M}_{t,t'}, \\[2ex]
h_{\tilde{g}_{t,t'}} & = & \tilde{v}^{-\frac{n}{n-2}} B_{\bar{g}_{t,t'}}\tilde{v} = \mu_1(t,t')\tilde{v}^{-\frac{2}{n-2}} & \text{on } \partial\bar{M}_{t,t'} = M_t \sqcup -M_{t'}.
\end{array}
\right.
$$

Here $\partial\bar{g}_{t,t'}$ is a restriction of the metric $\bar{g}_{t,t'}$ to the boundary $\partial\bar{M}_{t,t'}$. Again, we emphasize that the conformal metric $\tilde{g}_{t,t'}$ is scalar-flat and its mean curvature has a definite sign, the same as the principal eigenvalue $\mu_1(t,t')$. We choose the following normalization for the eigenfunctions $\tilde{v}_{t,t'}$:

$$
\tilde{v}_{t,t'}(x_{t,t'}) = 1. \tag{7.4}
$$

This construction provides a second family of Riemannian manifolds of corresponding scalar-flat satellites:

$$
\boldsymbol{\mathcal{W}}_\flat(M\times I, \bar{g}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \tilde{g}_{t,t'}, J_0, \bar{x}_{t,t'})\}_{0\leq t\leq t'\leq 1},
$$

determined by the manifold $(M\times I, \bar{g}, J_0, \bar{x})$. Here again, J_0 is the same interval as above, and the original metric $\bar{g} = g_t + dt^2$ is equidistant and ε_0-standard along the interval J_0. We emphasize that the metric $\tilde{g}_{t,t'}$ is scalar-flat, in particular, it is not ε_0-standard along J_0. We consider this construction as a map

$$
(M\times I, \bar{g}, J_0, \bar{x}) \mapsto \boldsymbol{\mathcal{W}}_\flat(M\times I, \bar{g}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \tilde{g}_{t,t'}, J_0, \bar{x}_{t,t'})\}_{0\leq t\leq t'\leq 1},
$$

generating a family of corresponding scalar-flat satellites.

7.3. **Minimal boundary problem.** Let $(W,\bar{g})$ be a compact Riemannian manifold with non-empty boundary $(\partial W, g)$, $\dim W = n$, as above in Sections 7.1 and 7.2.

To define the minimal boundary elliptic problem, we consider a relevant Rayleigh quotient and take the infimum:

$$\lambda_1 = \inf_{f\in C^\infty_+} \frac{\int_W (a_n|\nabla_{\bar g} f|^2 + R_{\bar g} f^2) d\sigma_{\bar g} + 2(n-1)\int_{\partial W} h_{\bar g} f^2 d\sigma_g}{\int_W f^2 d\sigma_{\bar g}}, \tag{7.5}$$

where $a_n = \frac{4(n-1)}{n-2}$ and $b_n = \frac{n-2}{2}$, as above. According to the standard elliptic theory, we obtain an elliptic boundary problem which will be denoted by $(L_{\bar g}, B_{\bar g})^\natural$, and the infimum λ_1 is the *principal eigenvalue of the boundary problem* $(L_{\bar g}, B_{\bar g})^\natural$. In paricular, there exists a smooth positive principal eigenfunction $\hat v$ minimizing the functional (7.5) which satisfies the corresponding Euler-Lagrange equations

$$\left\{ \begin{array}{lcll} L_{\bar g}\hat v & = & a_n\Delta_{\bar g}\hat v + R_{\bar g}\hat v = \lambda_1 \hat v & \text{on } W, \\ & & & \\ B_{\bar g}\hat v & = & \partial_\nu \hat v + b_n h_{\bar g}\hat v \equiv 0 & \text{on } \partial W. \end{array} \right. \tag{7.6}$$

Usually the eigenfunction $\hat v$ is normalized as $\int_W \hat v^2 d\sigma_{\bar g} = 1$, however, in the case of pointed manifolds we will use different normalization, see (7.7) below. Again, we will adjust this We notice that for the conformal metric $\hat g = \hat v^{\frac{4}{n-2}} \bar g$ we have:

$$\left\{ \begin{array}{lcll} R_{\hat g} & = & \hat v^{-\frac{n+2}{n-2}} L_{\bar g}\hat v \equiv \lambda_1 \hat v^{-\frac{4}{n-2}} & \text{on } W, \\ & & & \\ h_{\hat g} & = & \hat v^{-\frac{n}{n-2}} B_{\bar g}\hat v \equiv 0 & \text{on } \partial W. \end{array} \right.$$

Thus the conformal metric $\hat g$ has zero mean curvature and its scalar curvature $R_{\hat g}$ has a definite sign, the same as the principal eigenvalue λ_1.

Definition 7.2. We refer to the boundary problem $(L_{\bar g}, B_{\bar g})^\natural$, and the Euler-Lagrange equations (7.6) as the *minimal boundary problem on* $(W, \bar g)$, and we call the Riemannian manifold $(W, \hat g)$ the *minimal-boundary satellite of* $(W, \bar g)$.

7.4. **A family of minimal-boundary satellites $\boldsymbol{\mathcal{W}}_\natural$.** We return to our family of manifolds

$$\boldsymbol{\mathcal{W}}(M\times I, \bar g, J_0, \bar x) = \{(\bar M_{t,t'}, \bar g_{t,t'}, J_0, \bar x_{t,t'})\}.$$

For each pair (t,t'), $t\leq t'$, we consider the minimal-boundary satellite $(\bar M_{t,t'}, \hat g_{t,t'}, J_0, \bar x_{t,t'})$ of the manifold $(\bar M_{t,t'}, \bar g_{t,t'}, J_0, \bar x_{t,t'})$, where $\hat g_{t,t'} = \hat v_{t,t'}^{\frac{4}{n-2}} \bar g_{t,t'}$, and the eigenfunctions $\hat v_{t,t'}$ are normalized as:

$$\hat v_{t,t'}(x_{t,t'}) = 1. \tag{7.7}$$

Thus we obtain the third family of Riemannian manifolds, namely, the minimal-boundary satellites determined by the manifold $(M\times I, \bar g, J_0, \bar x)$:

$$\boldsymbol{\mathcal{W}}_\natural(M\times I, \bar g, J_0, \bar x) = \{(\bar M_{t,t'}, \hat g_{t,t'}, J_0, \bar x_{t,t'})\}_{0\leq t\leq t'\leq 1}. \tag{7.8}$$

We consider this construction as a map

$$(M \times I, \bar{g}, J_0, \bar{x}) \mapsto \boldsymbol{\mathcal{W}}_{\natural}(M \times I, \bar{g}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \hat{g}_{t,t'}, J_0, \bar{x}_{t,t'})\}_{0\leq t\leq t'\leq 1},$$

generating a family of minimal-boundary satellites $(\bar{M}_{t,t'}, \hat{g}_{t,t'}, J_0, \bar{x}_{t,t'})$ of the manifolds $(\bar{M}_{t,t'}, \bar{g}_{t,t'}, J_0, \bar{x})$. We refer to $\boldsymbol{\mathcal{W}}_{\natural}(M \times I, \bar{g}, J_0, \bar{x})$ as the *family of the minimal-boundary satellites of* $\boldsymbol{\mathcal{W}}(M \times I, \bar{g}, J_0, \bar{x})$.

Remark. Again, we emphasize that in general the metric $\hat{g}_{t,t'}$ is not ε_0-standard along J_0.

7.5. **Apriory bounds on the eigenvalues μ_1 and λ_1.** We would like to take a close look at the following family of manifolds:

$$\boldsymbol{\mathcal{W}}^{(q)} = \boldsymbol{\mathcal{W}}(M \times I, \bar{g}^{(q)}, J_0, \bar{x}) = \{(\bar{M}_{t,t'}, \bar{g}^{(q)}_{t,t'}, J_0, \bar{x}_{t,t'})\}.$$

We recall that the manifold $(M \times I, \bar{g}^{(q)}, J_0, \bar{x})$ is obtained by gluing in another $(q-1)$ Kobayashi tubes to the manifold $(M \times I, \bar{g}^{(1)}, J_0, \bar{x})$. First, we consider the families of the eigenvalues

$$\{\lambda_1(L_{\bar{g}^{(1)}_{t,t'}})\}_{0\leq t\leq t'\leq 1}, \qquad \{\mu_1(L_{\bar{g}^{(1)}_{t,t'}})\}_{0\leq t\leq t'\leq 1}. \tag{7.9}$$

Since both families are parametrized by a compact set $T = \{(t,t') \mid 0 \leq t \leq t' \leq 1\}$, there exist constants $\lambda_1^{(0)}$ and $\mu_1^{(0)}$ which bound the families (7.9) from below, i.e.,

$$\lambda_1(L_{\bar{g}^{(1)}_{t,t'}}) \geq \lambda_1^{(0)}, \qquad \mu_1(L_{\bar{g}^{(1)}_{t,t'}}) \geq \mu_1^{(0)} \quad \text{for all} \quad 0 \leq t \leq t' \leq 1.$$

Recall that the Kobayashi tube $(S^{n-1} \times I, \bar{h}, J_0', \bar{y}', J_0'', \bar{y}'')$ has positive eigenvalues $\lambda_1(L_{\bar{h}})$ and $\mu_1(L_{\bar{h}})$, see Lemma 6.9.

Lemma 7.3. *Let $k \geq 1$. Then $\lambda_1(L_{\bar{g}^{(q)}_{t,t'}}) \geq \lambda_1^{(0)}$, $\quad \mu_1(L_{\bar{g}^{(q)}_{t,t'}}) \geq \mu_1^{(0)}$ for all $0 \leq t \leq t' \leq 1$.*

Lemma 7.3 easily follows from Kobayshi inequality since the eigenvalues $\lambda_1(L_{\bar{h}})$ and $\mu_1(L_{\bar{h}})$ are positive.

We also would like to get some *apriori upper bounds* on the eigenvalues $\lambda_1(L_{\bar{g}^{(q)}_{t,t'}})$ and $\mu_1(L_{\bar{g}^{(q)}_{t,t'}})$.

Lemma 7.4. *There exists an integer $q > 1$ such that*

$$\lambda_1(L_{\bar{g}^{(q)}_{t,t'}}) \leq \lambda_1^{(1)}, \qquad \mu_1(L_{\bar{g}^{(q)}_{t,t'}}) \leq \mu_1^{(1)} \quad \textit{for all} \quad 0 \leq t \leq t' \leq 1,$$

where the constants $\lambda_1^{(1)}$ and $\mu_1^{(1)}$ depend only on the constant $\mathbf{c}^{(1)}$ and the Kobayashi tube

$$(S^{n-1} \times I, \bar{h}, J_0', \bar{y}', J_0'', \bar{y}'').$$

Proof. We prove it for $\lambda_1(L_{\bar{g}^{(q)}_{t,t'}})$. Recall that the manifold $(\bar{M}^{(q)}_{t,t'}, \bar{g}^{(q)}_{t,t'}, J_0, \bar{x}_{t,t'})$ is a result of gluing the original manifold $(\bar{M}^{(0)}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})$ with q copies of the Kobayashi tube $(S^{n-1} \times I, \bar{h}, J_0', \bar{y}', J_0'', \bar{y}'')$.

In particular, the volume $V^{(q)}_{t,t'}$ of the manifold $(\bar{M}^{(q)}_{t,t'}, \bar{g}^{(q)}_{t,t'}, J_0, \bar{x}_{t,t'})$ splits into the sum

$$V^{(q)}_{t,t'} = V_{t,t'} + q \cdot V_K,$$

where $V_{t,t'}$ is the volume of the manifold $(\bar{M}^{(0)}_{t,t'}, \bar{g}^{(0)}_{t,t'}, J_0, \bar{x}_{t,t'})$ and V_K is the volume of the Kobayshi tube $(S^{n-1} \times I, \bar{h}, J_0', \bar{y}', J_0'', \bar{y}'')$ with one standard part deleted, see Fig. 8. Similarly, the volume of boundary of $(\bar{M}^{(q)}_{t,t'}, \bar{g}^{(q)}_{t,t'}, J_0, \bar{x}_{t,t'})$ is a sum $V^{\partial}_{t,t'} + q \cdot V^{\partial}_K$, where $V^{\partial}_{t,t'}$ is the volume of the boundary of $(\bar{M}^{(0)}_{t,t'}, \bar{g}^{(0)}_{t,t'})$ and V_K is the volume of the boundary of the Kobayshi tube as above, see Fig. 8.

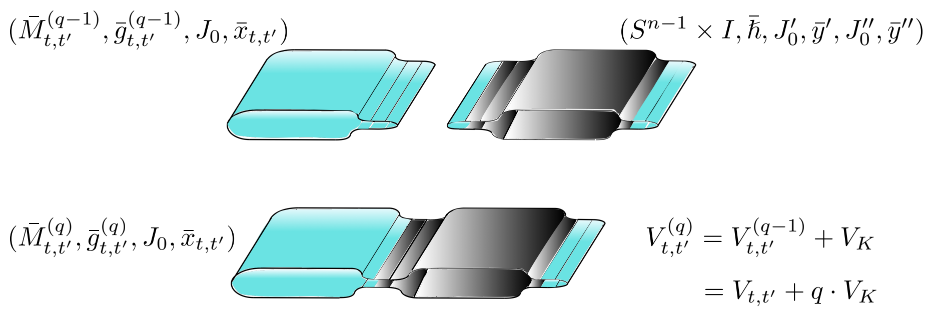

FIGURE 8. Gluing $(\bar{M}^{(q-1)}_{t,t'}, \bar{g}^{(q-1)}_{t,t'}, J_0, \bar{x}_{t,t'})$ with the Kobayashi tube: here the volume V_K is shown in the dark color.

By definition, we have

$$\lambda_1(L_{\bar{g}^{(q)}_{t,t'}}) = \inf_{f \in C^{\infty}_{+}} \frac{\int_{\bar{M}^{(q)}_{t,t'}} (a_n |\nabla_{\bar{g}^{(q)}_{t,t'}} f|^2 + R_{\bar{g}^{(q)}_{t,t'}} f^2) d\sigma_{\bar{g}^{(q)}_{t,t'}} + b_n \int_{\partial \bar{M}^{(q)}_{t,t'}} h_{\bar{g}^{(q)}_{t,t'}} f^2 d\sigma_{\partial \bar{g}^{(q)}_{t,t'}}}{\int_{\bar{M}^{(q)}_{t,t'}} f^2 d\sigma_{\bar{g}^{(q)}_{t,t'}}}$$

We choose $f = 1$ as a test function and denote by $R_0 = \max |R_{\bar{g}^{(q)}_{t,t'}}|$, and $h_0 = \max b_n |h_{\bar{g}^{(q)}_{t,t'}}|$:

$$\begin{aligned} \lambda_1(L_{\bar{g}^{(q)}_{t,t'}}) &\leq \frac{R_0 \int_{\bar{M}^{(q)}_{t,t'}} d\sigma_{\bar{g}^{(q)}_{t,t'}} + h_0 \int_{\partial \bar{M}^{(q)}_{t,t'}} d\sigma_{\partial \bar{g}^{(q)}_{t,t'}}}{\int_{\bar{M}^{(q)}_{t,t'}} d\sigma_{\bar{g}^{(q)}_{t,t'}}} \\ &= \frac{R_0 \cdot \mathrm{Vol}_{\bar{g}^{(q)}_{t,t'}}(\bar{M}^{(q)}_{t,t'}) + h_0 \cdot \mathrm{Vol}_{\partial \bar{g}^{(q)}_{t,t'}}(\partial \bar{M}^{(q)}_{t,t'})}{\mathrm{Vol}_{\bar{g}^{(q)}_{t,t'}}(\bar{M}^{(q)}_{t,t'})} \end{aligned}$$

Since $\mathrm{Vol}_{\bar{g}^{(q)}_{t,t'}}(\bar{M}^{(q)}_{t,t'}) = V_{t,t'} + q \cdot V_K$ and $\mathrm{Vol}_{\partial \bar{g}^{(q)}_{t,t'}}(\partial \bar{M}^{(q)}_{t,t'}) = V^{\partial}_{t,t'} + q \cdot V^{\partial}_K$, we obtain:

$$\begin{aligned} \lambda_1(L_{\bar{g}^{(q)}_{t,t'}}) &\leq \frac{R_0(V_{t,t'} + q \cdot V_K) + h_0(V^{\partial}_{t,t'} + q \cdot V^{\partial}_K)}{V_{t,t'} + q \cdot V_K} \\ &= \frac{\frac{1}{q}(R_0 V_{t,t'} + h_0 V^{\partial}_{t,t'}) + (R_0 V_K + h_0 V^{\partial}_K)}{\frac{1}{q} V_{t,t'} + V_K} \end{aligned}$$

Clearly, if $q\to\infty$, the right-hand side has the limit

$$\frac{R_0 V_K + h_0 V_K^{\partial}}{V_K} = R_0 + h_0\cdot \frac{V_K^{\partial}}{V_K}.$$

Thus there exists q such that

$$\lambda_1(L_{\bar g^{(q)}_{t,t'}}) \leq R_0 + h_0\cdot \frac{V_K^{\partial}}{V_K} + 1. \tag{7.10}$$

Since $\lambda_1(L_{\bar g^{(k)}_{t,t'}})$ is continuous on (t,t'), there exists q such that the inequality (7.10) holds for all (t,t'). The argument for $\mu_1(L_{\bar g^{(q)}_{t,t'}})$ is similar. $\square$

Remark. We denote $\boldsymbol{\nu} := \max\{|\lambda_1^{(0)}|, |\lambda_1^{(1)}|, |\mu_1^{(0)}|, |\mu_1^{(1)}|\}$. From now on, we assume that for each manifold $(\bar M_{t,t'}, \bar g^{(1)}_{t,t'}, J_0, \bar x_{t,t'})\in \boldsymbol{\mathcal{W}}^{(1)}$ the eigenvalues $\lambda_1(L_{\bar g^{(1)}_{t,t'}})$ and $\mu_1(L_{\bar g^{(1)}_{t,t'}})$ satisfy the bounds

$$|\lambda_1(L_{\bar g^{(1)}_{t,t'}})| \leq \boldsymbol{\nu}, \qquad |\mu_1(L_{\bar g^{(1)}_{t,t'}})| \leq \boldsymbol{\nu}. \tag{7.11}$$

Moreover, for the remaining part of the article, we assume the bounds (7.11) hold for the family $\boldsymbol{\mathcal{W}}^{(q)} = \boldsymbol{\mathcal{W}}(M\times I, \bar g^{(q)}, J_0, \bar x)$ for each $q\geq 1$. $\diamond$

8. Class of manifolds $\boldsymbol{\mathcal{O}}(M\times I, \bar g^{(1)}, J_0, \varepsilon_0, \mathbf{c}, k, \boldsymbol{\nu})$

8.1. **Conformal satellites and bounded geometry.** We start with the original counterexample, the manifold $(M\times I, \bar g^{(1)}, J_0, \bar x)$ which gives the family $\boldsymbol{\mathcal{W}}^{(1)} = \boldsymbol{\mathcal{W}}(M\times I, \bar g^{(1)}, J_0, \bar x)$. According to the above construction, we also have two more families

$$\begin{array}{rcccl} \boldsymbol{\mathcal{W}}^{(1)}_{\flat} & = & \boldsymbol{\mathcal{W}}_{\flat}(M\times I, \bar g^{(1)}, J_0) & = & \{(\bar M_{t,t'}, \tilde g^{(1)}_{t,t'}, J_0, \bar x_{t,t'})\}_{0\leq t\leq t'\leq 1} \\[2ex] \boldsymbol{\mathcal{W}}^{(1)}_{\natural} & = & \boldsymbol{\mathcal{W}}_{\natural}(M\times I, \bar g^{(1)}, J_0) & = & \{(\bar M_{t,t'}, \hat g^{(1)}_{t,t'}, J_0, \bar x_{t,t'})\}_{0\leq t\leq t'\leq 1} \end{array}$$

of scalar-flat and minimal boundary satellites.

Definition 8.1. Let $\boldsymbol{\mathcal{O}}(M\times I, \bar g^{(1)}, J_0, \varepsilon_0, \mathbf{c}, k, \boldsymbol{\nu})$ be a class of Riemannian manifolds $(M\times I, \bar g, J_0, \bar x)$ which are subjects of the following conditions:

(i) the metric $\bar g = g_t + dt^2$ is ε_0-standard along J_0, and $\lambda_1(L_{\bar g})\geq 0$;
(ii) the metric g_i is isotopic to the metric $g_i^{(1)}$ in the space $\mathcal{R}\mathrm{iem}^{\geq 0}(M)$, where $i=0,1$;
(iii) the eigenvalues $\mu_1(L_{\bar g_{t,t'}})$ and $\lambda_1(L_{\bar g_{t,t'}})$ satisfy the bounds:

$$|\mu_1(L_{\bar g_{t,t'}})| \leq \boldsymbol{\nu}, \qquad |\lambda_1(L_{\bar g_{t,t'}})| \leq \boldsymbol{\nu}$$

for each manifold $(\bar M_{t,t'}, \bar g_{t,t'}, J_0, \bar x_{t,t'})\in \boldsymbol{\mathcal{W}}(M\times I, \bar g, J_0, \bar x)$;
(iv) each manifold $(\bar M_{t,t'}, \bar g_{t,t'}, J_0, \bar x_{t,t'})\in \boldsymbol{\mathcal{W}}(M\times I, \bar g, J_0, \bar x)$ has $(\mathbf{c}, k)$-bounded geometry, where $\mathbf{c}\geq \mathbf{c}^{(1)}$, where the constant $\mathbf{c}^{(1)}$ is from Proposition 6.10, and $k\geq 8+2n$.

The next fact should be considered as a variation of well-known results, see [8] for the proof.

Proposition 8.2. *The class of manifolds $\boldsymbol{\mathcal{O}}(M \times I, \bar{g}^{(1)}, J_0, \varepsilon_0, \mathbf{c}, k, \boldsymbol{\nu})$ is compact with respect to the Gromov-Cheeger topology.*

8.2. **A kernel of a psc-concordance.** By assumption, there exists a pair $t < t'$ such that $\lambda_1(L_{\bar{g}_{t,t'}}) < 0$. We consider the following subset of the triangle T_0

$$\boldsymbol{\Xi} := \Lambda^{-1}((-\infty,0)) = \{ \ (t,t') \mid t < t', \quad \lambda_1(L_{\bar{g}_{t,t'}}) < 0 \ \} \subset T_0.$$

The set $\boldsymbol{\Xi}$ is an open subset of T_0. We denote by $\bar{\boldsymbol{\Xi}} \subset I^2$ its closure, and by $\partial\boldsymbol{\Xi} = \bar{\boldsymbol{\Xi}} \setminus \boldsymbol{\Xi}$ its boundary. Since the function $\Lambda : (t,t') \mapsto \lambda_1(L_{\bar{g}_{t,t'}})$ is continuous, we see that $\Lambda(t,t') = 0$ if $(t,t') \in \partial\boldsymbol{\Xi}$. However, in general, $\partial\boldsymbol{\Xi} \neq \Lambda^{-1}(0)$.

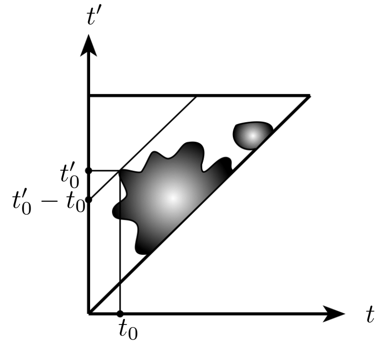

FIGURE 9. The size of the kernel $\iota(M \times I, \bar{g}, J_0, \bar{x})$

We set $\iota(M \times I, \bar{g}, J_0, \bar{x}) := \sup\{ \ |t' - t| \mid (t,t') \in \Xi \ \}$. Continuity implies that there exists a pair $(t_0, t'_0) \in \partial\boldsymbol{\Xi}$ such that $\iota(M \times I, \bar{g}, \bar{x}) = t'_0 - t_0 > 0$.

Definition 8.3. Let $(t_0, t'_0) \in \partial\boldsymbol{\Xi}$ be a pair such that $\iota(M \times I, \bar{g}, \bar{x}) = t'_0 - t_0 > 0$. Then the Riemannian manifold $(\bar{M}_{t_0,t'_0}, \bar{g}_{t_0,t'_0}, J_0, \bar{x}_{t_0,t'_0})$ is a *kernel of the concordance* $(M \times I, \bar{g}, J_0, \bar{x})$ and the number $\iota(M \times I, \bar{g}, J_0, \bar{x})$ is called the *size of a kernel*. We write $\iota := \iota(M \times I, \bar{g}, J_0, \bar{x})$ if it is clear which psc-concordance do we use.

We notice that a kernel $(\bar{M}_{t_0,t'_0}, \bar{g}_{t_0,t'_0}, J_0, \bar{x}_{t_0,t'_0})$ could be not unique, however the size of a kernel $\iota(M \times I, \bar{g}, J_0, \bar{x})$ is uniquely defined.

Let $(M \times I, \bar{g}, J_0, \bar{x}) \in \boldsymbol{\mathcal{O}}(M \times I, \bar{g}^{(1)}, J_0, \varepsilon_0, \mathbf{c}, k, \boldsymbol{\nu})$. Then by assumption, $\iota(M \times I, \bar{g}, J_0, \bar{x}_0) > 0$: otherwise we have a contradiction which would prove Theorem 2.9. We consider the invariant

$$\iota_0 := \inf\{ \ \iota(M \times I, \bar{g}, J_0, \bar{x}) \mid (M \times I, \bar{g}, J_0, \bar{x}) \in \boldsymbol{\mathcal{O}}(M \times I, \bar{g}^{(1)}, J_0, \varepsilon_0, \mathbf{c}, k, \boldsymbol{\nu}) \ \}.$$

To complete the proof, we have to analyze two cases:

(1) $\iota_0 > 0$;

(2) $\iota_0 = 0$.

These two cases require different strategies. Now we need one more technical section.

8.3. **Satellite manifolds and Cheeger-Gromov convergence.** Let $\{(W^{(i)},\bar{g}^{(i)},\bar{x}^{(i)})\}$ be a sequence of compact Riemannian manifolds with non-empty boundary as above. We assume that each manifold $(W^{(i)},\bar{g}^{(i)},\bar{x}^{(i)})$ has $(\mathbf{c},k)$-bounded geometry. By compactness, the sequence $\{(W^{(i)},\bar{g}^{(i)},\bar{x}^{(i)})\}$ contains a convergent subsequence. Thus we assume that the sequence $\{(W^{(i)},\bar{g}^{(i)},\bar{x}^{(i)})\}$ is already convergent, i.e.,

$$\lim_{i\to\infty}(W^{(i)},\bar{g}^{(i)},\bar{x}^{(i)})=(W^{(\infty)},\bar{g}^{(\infty)},\bar{x}^{(\infty)}),$$

where the limiting manifold $(W^{(\infty)},\bar{g}^{(\infty)},\bar{x}^{(\infty)})$ also has $(\mathbf{c},k)$-bounded geometry. It is important that the limiting manifold could be non-compact.

Now we consider corresponding satellites sequences

$$\{(W^{(i)},\tilde{g}^{(i)},\bar{x}^{(i)})\},\quad \{(W^{(i)},\hat{g}^{(i)},\bar{x}^{(i)})\}$$

of the scalar-flat and minimal-boundary satellites respectively, i.e., $\tilde{g}^{(i)}=\tilde{v}_i^{\frac{4}{n-2}}\bar{g}^{(i)}$, $\hat{g}^{(i)}=\hat{v}_i^{\frac{4}{n-2}}\bar{g}^{(i)}$, where $\tilde{v}_i$ and $\hat{v}_i$ are solutions of the corresponding Euler-Lagrange equations (7.2) and (7.6), are normalized as follows:

$$\tilde{v}_i(\bar{x}^{(i)})=1,\quad \hat{v}_i(\bar{x}^{(i)})=1. \tag{8.1}$$

The following result follows directly from [8, Theorems A, B, C], where the technique of conformal satellites has been developed.

Theorem 8.4. (B. Botvinnik, O. Müller, [8, Theorems A, B, C]) *Let $\{(W^{(i)},\bar{g}^{(i)},\bar{x}^{(i)})\}$ be a sequence of compact pointed Riemannian manifolds with non-empty boundaries, $\dim W^{(i)}=n$. Assume that*

(i) *each manifold $(W^{(i)},\bar{g}^{(i)},\bar{x}^{(i)})$ has $(\mathbf{c},k)$-bounded geometry for some $\mathbf{c}>0$ and $k\geq 8+2n$;*
(ii) *the principal eigenvalues $\mu_1(L_{\bar{g}^{(i)}})$ and $\lambda_1(L_{\bar{g}^{(i)}})$ are bounded, i.e.,*

$$|\mu_1(L_{\bar{g}^{(i)}})|\leq\boldsymbol{\nu},\quad |\lambda_1(L_{\bar{g}^{(i)}})|\leq\boldsymbol{\nu}$$

for some constant $\boldsymbol{\nu}>0$.

Then there exists a converging subsequence $\{(W^{(i_k)},\bar{g}^{(i_k)},\bar{x}^{(i_k)})\}$ of the sequence $\{(W^{(i)},\bar{g}^{(i)},\bar{x}^{(i)})\}$, such that

(a) *the subsequences of the satellites*

$$\{(W^{(i_k)},\tilde{g}^{(i_k)},\bar{x}^{(i_k)})\}\quad\text{and}\quad\{(W^{(i_k)},\hat{g}^{(i_k)},\bar{x}^{(i_k)})\},$$

where $\tilde{g}^{(i_k)}=\tilde{v}_{i_k}^{\frac{4}{n-2}}\bar{g}^{(i_k)}$, $\hat{g}^{(i_k)}=\hat{v}_{i_k}^{\frac{4}{n-2}}\bar{g}^{(i_k)}$, and the solutions $\tilde{v}_{i_k}$ and $\hat{v}_{i_k}$ of the scalar-flat and minimal-boundary satellite problems are normalized as in (8.1), are also convergent and have $(\mathbf{c}',k-5-2n)$-bounded geometry for some $\mathbf{c}'>0$;

(b) *the limiting manifolds*

$$\begin{array}{rcl}
(\bar{W}^{(\infty)}, \bar{g}^{(\infty)}, \bar{x}^{(\infty)}) & := & \lim\limits_{k\to\infty} (W^{(i_k)}, \bar{g}^{(i_k)}, \bar{x}^{(i_k)}) \\[2ex]
(\tilde{W}^{(\infty)}, \tilde{g}^{(\infty)}, \tilde{x}^{(\infty)}) & := & \lim\limits_{k\to\infty} (W^{(i_k)}, \tilde{g}^{(i_k)}, \bar{x}^{(i_k)}) \\[2ex]
(\hat{W}^{(\infty)}, \hat{g}^{(\infty)}, \hat{x}^{(\infty)}) & := & \lim\limits_{k\to\infty} (W^{(i_k)}, \hat{g}^{(i_k)}, \bar{x}^{(i_k)})
\end{array}$$

are diffeomorphic via diffeomorphisms of pointed manifolds

$$\tilde{\varphi} : (\tilde{W}^{(\infty)}, \tilde{x}^{(\infty)}) \to (\bar{W}^{(\infty)}, \bar{x}^{(\infty)}), \quad \hat{\varphi} : (\hat{W}^{(\infty)}, \hat{x}^{(\infty)}) \to (\bar{W}^{(\infty)}, \bar{x}^{(\infty)}),$$

such that the metrics $\tilde{\varphi}^*\tilde{g}^{(\infty)}$ *and* $\hat{\varphi}^*\tilde{g}^{(\infty)}$ *are conformal to* $\bar{g}^{(\infty)}$.

9. Proof of Theorem 2.9: Case (1)

9.1. **Taking the limits.** We find a sequence of manifolds $(M \times I, \bar{g}^{(j)}, J_0, \bar{x}_j)$ and a sequence of corresponding parameters $t_j < t'_j$ such that $\iota(M \times I, \bar{g}^{(j)}, J_0, \bar{x}_j) = t'_j - t_j$ and

$$\lim_{j\to\infty} \iota(M \times I, \bar{g}^{(j)}, J_0, \bar{x}_j) = \iota_0.$$

We consider a sequence of corresponding kernels $\{(\bar{M}_{t_j,t'_j}, \bar{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}$, and then choose a converging subsequence of manifolds $\{(M \times I, \bar{g}^{(j)}, J_0, \bar{x}_j)\}$, and we pass to a subsequence so that the corresponding subsequence of kernels $\{(\bar{M}_{t_j,t'_j}, \bar{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}$ is also converging. We also consider corresponding sequences of scalar-flat and minimal-boundary satellites

$$\{(M \times I, \tilde{g}^{(j)}, J_0, \bar{x}_j)\} \quad \text{and} \quad \{(\bar{M}_{t_j,t'_j}, \tilde{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\},$$

$$\{(M \times I, \hat{g}^{(j)}, J_0, \bar{x}_j)\} \quad \text{and} \quad \{(\bar{M}_{t_j,t'_j}, \hat{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}.$$

Remark. We distinguish here the satellites $(\bar{M}_{t_j,t'_j}, \tilde{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})$ and $(\bar{M}_{t_j,t'_j}, \hat{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})$ of the kernel $(\bar{M}_{t_j,t'_j}, \bar{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})$, however here $\lambda_1(L_{\bar{g}^{(j)}_{t_j,t'_j}}) = \mu_1(L_{\bar{g}^{(j)}_{t_j,t'_j}}) = 0$. Hence the scalar-flat and the minimal boundary problems are the same here, and these satellites coincide.

We use Theorem 8.4 to pass to subsequences so that all six sequences

$$\begin{array}{ll}
\{(M \times I, \bar{g}^{(j)}, J_0, \bar{x}_j)\}, & \{(\bar{M}_{t_j,t'_j}, \bar{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}, \\[2ex]
\{(M \times I, \tilde{g}^{(j)}, J_0, \bar{x}_j)\}, & \{(\bar{M}_{t_j,t'_j}, \tilde{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}, \\[2ex]
\{(M \times I, \hat{g}^{(j)}, J_0, \bar{x}_j)\}, & \{(\bar{M}_{t_j,t'_j}, \hat{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}
\end{array}$$

are convergent. Now we take a limit of the manifolds $(M\times I, \bar g^{(j)}, J_0, \bar x_j)$ and their scalar-flat and minimal-boundary satellites:

$$
\begin{array}{rcl}
\lim\limits_{j\to\infty}(M\times I, \bar g^{(j)}, J_0, \bar x_j) & = & (\bar Z^{(\infty)}, \bar g^{(\infty)}, J_0, \bar x_\infty),\\[2ex]
\lim\limits_{j\to\infty}(M\times I, \tilde g^{(j)}, J_0, \bar x_j) & = & (\tilde Z^{(\infty)}, \tilde g^{(\infty)}, J_0, \bar x_\infty),\\[2ex]
\lim\limits_{j\to\infty}(M\times I, \hat g^{(j)}, J_0, \bar x_j) & = & (\hat Z^{(\infty)}, \hat g^{(\infty)}, J_0, \bar x_\infty).
\end{array}
\tag{9.1}
$$

We also take limits of corresponding kernels and their scalar-flat and minimal-boundary satellites:

$$
\begin{array}{rcl}
\lim\limits_{j\to\infty}(\bar M_{t_j,t_j'}, \bar g^{(j)}_{t_j,t_j'}, J_0, \bar x_{t_j,t_j'}) & = & (\bar X^{(\infty)}_{t_\infty,t'_\infty}, \bar g^{(\infty)}_{t_\infty,t'_\infty}, J_0, \bar x_{t_\infty,t'_\infty}),\\[2ex]
\lim\limits_{j\to\infty}(\bar M_{t_j,t_j'}, \tilde g^{(j)}_{t_j,t_j'}, J_0, \bar x_{t_j,t_j'}) & = & (\tilde X^{(\infty)}_{t_\infty,t'_\infty}, \tilde g^{(\infty)}_{t_\infty,t'_\infty}, J_0, \bar x_{t_\infty,t'_\infty}),\\[2ex]
\lim\limits_{j\to\infty}(\bar M_{t_j,t_j'}, \hat g^{(j)}_{t_j,t_j'}, J_0, \bar x_{t_j,t_j'}) & = & (\hat X^{(\infty)}_{t_\infty,t'_\infty}, \hat g^{(\infty)}_{t_\infty,t'_\infty}, J_0, \bar x_{t_\infty,t'_\infty}).
\end{array}
\tag{9.2}
$$

By construction, all the manifolds $(\bar M_{t_j,t_j'}, \tilde g^{(j)}_{t_j,t_j'}, J_0, \bar x_{t_j,t_j'})$ and $(\bar M_{t_j,t_j'}, \hat g^{(j)}_{t_j,t_j'}, J_0, \bar x_{t_j,t_j'})$ are scalar-flat. Thus the corresponding limiting manifolds

$$
(\tilde X^{(\infty)}_{t_\infty,t'_\infty}, \tilde g^{(\infty)}_{t_\infty,t'_\infty}, J_0, \bar x_{t_\infty,t'_\infty}) \quad\text{and}\quad (\hat X^{(\infty)}_{t_\infty,t'_\infty}, \hat g^{(\infty)}_{t_\infty,t'_\infty}, J_0, \bar x_{t_\infty,t'_\infty})
$$

are scalar-flat as well. By Theorem 8.4, there exist diffeomorphisms

$$
\tilde\varphi : (\tilde X^{(\infty)}_{t_\infty,t'_\infty}, \tilde x^{(\infty)}_{t_\infty,t'_\infty}) \longrightarrow (\bar X^{(\infty)}_{t_\infty,t'_\infty}, \bar x^{(\infty)}_{t_\infty,t'_\infty}), \quad \hat\varphi : (\hat X^{(\infty)}_{t_\infty,t'_\infty}, \hat x^{(\infty)}_{t_\infty,t'_\infty}) \longrightarrow (\bar X^{(\infty)}_{t_\infty,t'_\infty}, \bar x^{(\infty)}_{t_\infty,t'_\infty})
$$

such that the metrics $\tilde\varphi^*\tilde g^{(\infty)}_{t_\infty,t'_\infty}$ and $\hat\varphi^*\tilde g^{(\infty)}_{t_\infty,t'_\infty}$ are conformal to the metric $\bar g^{(\infty)}_{t_\infty,t'_\infty}$.

Recall that we are considering the case when $\iota_0 = t'_\infty - t_\infty > 0$. Now we attach the Kobayashi tube

$$
(\bar K^{(1)}, \bar h^{(1)}, J_0', \bar y', J_0'', \bar y'') := (S^{n-1}\times I, \bar h, J_0', \bar y', J_0'', \bar y'') \tag{9.3}
$$

to each manifold from the sequences

$$
\{(M\times I, \bar g^{(j)}, J_0, \bar x_j)\} \quad\text{and}\quad \{(\bar M_{t_j,t_j'}, \bar g^{(j)}_{t_j,t_j'}, J_0, \bar x_{t_j,t_j'})\}
$$

as well as to the limiting manifolds

$$
(\bar Z^{(\infty)}, \bar g^{(\infty)}, J_0, \bar x_\infty) \quad\text{and}\quad (\bar X^{(\infty)}_{t_\infty,t'_\infty}, \bar g^{(\infty)}_{t_\infty,t'_\infty}, J_0, \bar x_{t_\infty,t'_\infty}) .
$$

We do this according to the identification given above (6.4). We denote the resulting sequences as

$$
\{((M\times I)^{(1)}, \bar{\mathsf g}^{(j,1)}, J_0, \bar x_j)\} \quad\text{and}\quad \{(\bar M^{(1)}_{t_j,t_j'}, \bar{\mathsf g}^{(j,1)}_{t_j,t_j'}, J_0, \bar x_{t_j,t_j'})\} . \tag{9.4}
$$

Here the extra index "1" indicates that we attached one copy of the Kobayashi tube.

9.2. **First key observation.** According to Proposition 11.1, each manifold $(\bar{M}^{(1)}_{t_j,t'_j}, \bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})$ from the sequence $\{(\bar{M}^{(1)}_{t_j,t'_j}, \bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}$ is such that $\mu_1(L_{\bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}}) > 0$. Thus $\lambda_1(L_{\bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}}) > 0$ for each j. Then we consider the minimal-boundary satellite sequences

$$\{((M\times I)^{(1)}, \hat{\mathrm{g}}^{(j,1)}, J_0, \bar{x}_j)\} \quad \text{and} \quad \{(\bar{M}^{(1)}_{t_j,t'_j}, \hat{\mathrm{g}}^{(j,1)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}. \tag{9.5}$$

Since $\lambda_1(L_{\bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}}) > 0$, we obtain that $R_{\hat{\mathrm{g}}^{(j,1)}_{t_j,t'_j}} > 0$ for all j. By passing to subsequences, we may assume that all four sequences (9.5) and (9.4) are converging.

9.3. **Second key observation.** Clearly we may pass to converging subsequences in (9.5). Then we obtain that

$$\lim_{j\to\infty} \lambda_1(L_{\bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}}) \geq 0.$$

If this limit is strictly positive we would be done. To get more control on the geometry of the limiting manifolds, we need one more geometrical construction.

First, for each $j=1,2,\ldots$, we find an eigenfunction $\hat{v}_{t_j,t'_j}$ corresponding to the eigenvalue $\lambda_1(L_{\bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}})$. We assume that $\hat{v}_{t_j,t'_j}(\bar{x}_{t_j,t'_j}) = 1$. Then for the metric $\hat{\mathrm{g}}^{(j,1)}_{t_j,t'_j} = v^{\frac{4}{n-2}}_{t_j,t'_j} \cdot \bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}$ has the scalar curvature

$$R_{\hat{\mathrm{g}}^{(j,1)}_{t_j,t'_j}} = \lambda_1(L_{\bar{\mathrm{g}}^{(j,1)}_{t_j,t'_j}}) v^{-\frac{4}{n-2}}_{t_j,t'_j} > 0.$$

Now for each $j=1,2,\ldots$, we attach the one more Kobayashi tube, we denote the results as:

$$((M\times I)^{(2)}, \hat{\mathrm{g}}^{(j,2)}, J_0, \bar{x}_j) \quad \text{and} \quad (\bar{M}^{(2)}_{t_j,t'_j}, \hat{\mathrm{g}}^{(j,2)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})$$

Recall that in order to construct $((M\times I)^{(2)}, \hat{\mathrm{g}}^{(j,2)}, J_0, \bar{x}_j)$ (respectively, $(\bar{M}^{(2)}_{t_j,t'_j}, \hat{\mathrm{g}}^{(j,2)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})$), we glue the manifold $((M\times I)^{(1)}, \hat{\mathrm{g}}^{(j,1)}, J_0, \bar{x}_j)$ (respectively, $(\bar{M}^{(1)}_{t_j,t'_j}, \hat{\mathrm{g}}^{(j,1)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})$) and the Kobayashi tube

$$(\bar{K}^{(1)}, \bar{h}^{(1)}, J'_0, \bar{y}', J''_0, \bar{y}'')$$

(the same for both cases) along the cylinder

$$(U, \breve{g}) := (S^{n-2}\times[0,a_0]\times J'_0, \bar{g}_o^{(n-2)} + ds^2 + dt^2)$$

Let $\tilde{\mathrm{g}}^{(j,1)} = v^{\frac{4}{n-2}}_{t_j,t'_j} \hat{\mathrm{g}}^{(j,1)}_{t_j,t'_j}$ be the corresponding conformal metric on $\bar{M}^{(1)}_{t_j,t'_j}$.

Then we use Proposition 11.7 to glue together the metrics $\tilde{\mathrm{g}}^{(j,1)}$ and $\bar{h}^{(1)}$ to obtain new metric $\tilde{\mathrm{g}}^{(j,2)}$ on $\bar{M}^{(2)}_{t_j,t'_j}$ such that

$$\tilde{\mathrm{g}}^{(j,2)} = \begin{cases} \tilde{\mathrm{g}}^{(j,2)} & \text{on } \bar{M}^{(1)}_{t_j,t'_j} \setminus U \\ \bar{h}^{(1)} & \text{on } K^{(1)} \setminus U. \end{cases}$$

with zero mean curvature and bounded injectivity radius. In particular, we have that for the restriction of $\tilde{\mathrm{g}}^{(j,2)}$ to $K^{(1)} \setminus U$, the scalar curvature $R_{\tilde{\mathrm{g}}^{(j,2)}}$ is bounded from below by $R_1 > 0$, where $R_1 = \min R_{\bar{h}^{(1)}}$.

Now we pass to converging subsequences (and we do not change the notation), so that both sequences

$$\{((M\times I)^{(2)}, \hat{\mathrm{g}}^{(j,2)}, J_0, \bar{x}_j)\} \quad \text{and} \quad \{(\bar{M}^{(2)}_{t_j,t'_j}, \tilde{\mathrm{g}}^{(j,2)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}$$

smoothly converge:

$$\begin{aligned} \lim_{j\to\infty}((M\times I)^{(2)}, \hat{\mathrm{g}}^{(j,2)}, J_0, \bar{x}_j) &= (\hat{Z}^{(\infty)}, \hat{\mathrm{g}}^{(\infty,2)}, J_0, \bar{x}_\infty) \\ \lim_{j\to\infty}(\bar{M}^{(2)}_{t_j,t'_j}, \tilde{\mathrm{g}}^{(j,2)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j}) &= (\hat{X}^{(\infty)}_{t_\infty,t'_\infty}, \tilde{\mathrm{g}}^{(\infty,2)}_{t_\infty,t'_\infty}, J_0, \bar{x}_\infty). \end{aligned} \tag{9.6}$$

As the result of the last construction, all manifolds in the sequence $\{(\bar{M}^{(2)}_{t_j,t'_j}, \tilde{\mathrm{g}}^{(j,2)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}$ have standard Kobayashi metric on $K^{(1)} \setminus U$. Hence the scalar curvature $R_{\tilde{\mathrm{g}}^{(j,2)}_{t_j,t'_j}}$ is bounded by $R_1 > 0$ on $K^{(1)} \setminus U$. Clearly $\lambda_1(L_{\mathrm{g}^{(j,2)}_{t_j,t'_j}}) > 0$ for each $j = 1, 2, \ldots$. Hence the assumption

$$\lim_{j\to\infty} \lambda_1(L_{\mathrm{g}^{(j,2)}_{t_j,t'_j}}) = 0$$

contradicts to the condition that $R_{\tilde{\mathrm{g}}^{(j,2)}_{t_\infty,t'_\infty}} \geq R_1 > 0$.

10. Proof of Theorem 2.9: Case (2)

10.1. **Again, taking appropriate limits.** Now we assume that $\iota_0 = 0$, and we find a sequence of manifolds $(M\times I, \bar{g}^{(j)}, J_0, \bar{x}_j)$ and a sequence of corresponding parameters $t_j < t'_j$ such that $\iota(M\times I, \bar{g}^{(j)}, J_0, \bar{x}_j) = t'_j - t_j$ and

$$\lim_{j\to\infty} \iota(M\times I, \bar{g}^{(j)}, J_0, \bar{x}_j) = 0.$$

Then we consider a sequence of corresponding kernels $\{(\bar{M}_{t_j,t'_j}, \bar{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}$, and then choose a converging subsequence of manifolds $\{(M\times I, \bar{g}^{(j)}, J_0, \bar{x}_j)\}$, and then pass to a subsequence so that the corresponding subsequence of kernels $\{(\bar{M}_{t_j,t'_j}, \bar{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}$ is also converging.

Since we have that $t'_j - t_j \to 0$ as $j\to\infty$, the metric $\bar{g}^{(j)}_{t_j,t'_j}$ on $M_{t_j,t'_j} \cong M\times[0,1]$ is almost cylindrical. Then we choose $t^{(j)}_\bullet$ to be the middle of the interval $[t_j, t'_j]$, and consider the slice $(M^{(j)}_\bullet, g^{(j)}_\bullet)$ where $M^{(j)}_\bullet = M\times\{t^{(j)}_\bullet\}$ and $g^{(j)}_\bullet$ is the restriction of $\bar{g}^{(j)}_{t_j,t'_j}$ to $M^{(j)}_\bullet$. Recall that $[t_j, t'_j] \subset [0,1]$ is an interval of maximal size such that $\lambda_1(L_{\bar{g}^{(j)}_{t_j,t'_j}}) = 0$.

Now, by passing to a subsequence of the kernels $\{(\bar{M}_{t_j,t'_j}, \bar{g}^{(j)}_{t_j,t'_j}, J_0, \bar{x}_{t_j,t'_j})\}$, we can assume that the sequence of closed manifolds $\{(M^{(j)}_\bullet, g^{(j)}_\bullet, \bar{x}^{(j)}_\bullet)\}$, where $\bar{x}^{(j)}_\bullet = \bar{x}_{t_j,t'_j}$ (we recall that we always

choose $\bar{x}_{t_j,t'_j}$ at the middle of the corresponding interval) is also convergent. Let

$$\lim_{j\to\infty}(M_{\bullet}^{(j)}, g_{\bullet}^{(j)}, \bar{x}_{\bullet}^{(j)}) = (Y_{\bullet}^{(\infty)}, g_{\bullet}^{(\infty)}, \bar{x}_{\bullet}^{(\infty)}).$$

Lemma 10.1. *Let* $L_{g_{\bullet}^{(j)}}$ *be as above. Then* $\lim_{j\to\infty}\lambda_1(L_{g_{\bullet}^{(j)}}) = 0$.

Recall that for given j, the function $t \mapsto \lambda_1(L_{g_t^{(j)}})$ is a continuous function on $[0,1]$. Thus Lemma 10.1 implies the following

Lemma 10.2. *For the family of Riemannian manifolds* $\{(M_{\bullet}^{(j)}, g_{\bullet}^{(j)}, x_t)\}$ *and any* $\vartheta > 0$, *there exists* j_0 *such that* $|\lambda_1(L_{g_t^{(j)}})| < \vartheta$ *for all* $j > j_0$ *and* $t \in K^{(j)}$.

Remark. We note that if $\lambda_1(L_{g_t^{(j)}}) \geq 0$ for some j and all $t \in [t_j, t'_j]$, we would get a contradiction; hence we have to deal with the case when $\lambda_1(L_{g_t^{(j)}}) < 0$.

10.2. **Few words on the Ricci Flow.** We take a pause to recall few necessary facts about Ricci flow. Let (N,h) be a closed compact Riemannain manifold. We consider the Ricci flow:

$$\frac{\partial h(\tau)}{\partial \tau} = -2\,\mathrm{Ric}_{h(\tau)}, \qquad h(0) = h. \tag{10.1}$$

Let $\lambda_1(L_{h(\tau)})$ be the principal eigenvalue of the conformal Laplacian, and $f = f(\tau)$ be the corresponding eigenfunction normalized as

$$\int_N f^2 d\sigma_{h(\tau)} = 1.$$

Let $\varphi = \varphi(\tau)$ be such that $e^{-\varphi} = f^2$, and $R_{ij} = R_{ij}(\tau)$ denotes the curvature of $h(\tau)$. Then we have the following formula proved by X. Cao, see [11, Theorem 1.5]:

$$\frac{d\lambda_1(L_{h(\tau)})}{d\tau} \;=\; \frac{1}{2}\int_N |R_{ij} + \nabla_i\nabla_j\varphi|^2 e^{-\varphi} d\sigma_{h(\tau)} + \frac{1}{2(n-2)}\int_N |\,\mathrm{Ric}_{h(\tau)}\,|^2 e^{-\varphi} d\sigma_{h(\tau)} \tag{10.2}$$

for all $0 < \tau < T_0$, where it is assumed that the Ricci Flow (10.1) exists for all $\tau \in [0, T_0)$.

10.3. **Back to the Case 2.** Let $\{t_j^*\}$ be any sequence such that $t_j^* \in [t_j, t'_j]$. We denote by g_j the metric $g_{t_j^*}^{(j)}$, and let $M_{t_j^*} := M \times \{t_j^*\}$. Now we recall that every slice $(M_{t_j^*}, g_j, x_{t_j^*})$ has a standard torpedo part, namely it contains the "torpedo submanifold" $\bar{U}$

$$\begin{array}{c} (D^{n-1}\times I, g_{\mathrm{torp}}^{(n-1)}(\varepsilon_0)) = (S^{n-2}(\varepsilon)\times[0,a_0]\times I, g_o^{(n-2)} + ds^2) \\ \cup (S^{n-2}\times[a_0,a_1]\times I, \bar{g}_o) \cup (S_+^{n-1}(\varepsilon_0)\times I, g_o^{(n-1)}), \end{array} \tag{10.3}$$

where, we recall, $(S^{n-2}(\varepsilon_0)\times[0,a_0], g_o^{(n-2)} + ds^2)$ is the standard cylinder, $(S_+^{n-1}(\varepsilon_0), g_o^{(n-1)})$ is a standard hemisphere of radius ε_0, and $(S^{n-2}(\varepsilon)\times[a_0,a_1], \bar{g}_o)$ is a transition region between the standard pieces. We denote the torpedo submanifold (10.3) by $\bar{U}$. We observe the following fact:

Lemma 10.3. *Let $\{t_j^*\}$ be any sequence such that $t_j^* \in [t_j, t_j']$, and $(M_{t_j^*}, g_{t_j^*}, x_{t_j^*})$ be a slice as above. There exists a constant $C(n, \varepsilon_0)$ which depends only on n and ε_0 such that*

$$\int_{M_{t_j^*}} |\operatorname{Ric}_{g_{t_j^*}}| \, d\sigma_{g_{t_j^*}} > C(n, \varepsilon_0). \tag{10.4}$$

Indeed, the integral in (10.4) is bounded from below by restricting the norm $|\operatorname{Ric}_{g_{t_j^*}}|$ to the standard torpedo manifold $\bar{U}$, given by (10.3), which is the same for all slices $(M_{t_j^*}, g_{t_j^*}, x_{t_j^*})$.

Now, for each slice $(M_{t_j^*}, g_{t_j^*}, x_{t_j^*})$ as above, we consider the corresponding conformal Laplacian $L_{g_{t_j^*}}$, the principal eigenvalue $\lambda_1(L_{g_{t_j^*}})$ and corresponding eigenfunction f_j normalized in the standard way, i.e.

$$\int_{M\times\{t_j\}} f_j^2 d\sigma_{g_{t_j^*}} = 1. \tag{10.5}$$

We denote by m_j the following minimizing constant:

$$m_j = \min\{ \ f_j^2(x) \mid x \in \bar{U}, \ t_j^* \in [t_j, t_j'] \ \}.$$

Since $[t_j, t_j']$ are compact intervals, $m_j > 0$ for each j. Compactness implies the following fact:

Lemma 10.4. *There exists $m_\infty > 0$ such that $m_j \geq m_\infty$.*

Clearly, Lemma 10.3 and Lemma 10.4 imply the following:

Lemma 10.5. *There exists a constant $C_1(n, \varepsilon_0) > 0$ such that*

$$\int_{M_{t_j^*}} |\operatorname{Ric}_{g_{t_j^*}}|^2 f_j^2 d\sigma_{g_{t_j^*}} \geq C_1(n, \varepsilon_0) > 0, \tag{10.6}$$

where $C_1(n, \varepsilon_0)$ does not depend on a choice of the sequence $\{t_j^*\}$, $t_j^* \in [t_j, t_j']$.

Now we consider the limiting manifold $(Y_\bullet^{(\infty)}, g_\bullet^{(\infty)}, \bar{x}_\bullet^{(\infty)})$, which is, in general, non-compact smooth manifold with bounded geometry. According to Shi's estimates, see [32] and [33], there exists a constant $T(n-1, \mathbf{K}_q^{(1)})$, where $\mathbf{K}_q^{(1)}$ is the bound for our class of manifolds, such that the Ricci Flow exists for the times in the interval $[0, T(n-1, \mathbf{K}_q^{(1)})$. We start Ricci flow on $(Y_\bullet^{(\infty)}, g_\bullet^{(\infty)}, \bar{x}_\bullet^{(\infty)})$, and also we have that

$$\frac{d\lambda_1(L_{g_\bullet^{(\infty)}(\tau)})}{d\tau} \geq C_1(n, \varepsilon_0) > 0.$$

Thus there exists $\tau_0 < T(n-1, \mathbf{K}_q^{(1)})$ such that $\lambda_1(L_{g_\bullet^{(\infty)}(\tau_0)}) > 0$. Now we start Ricci flow on each manifold $(M_{t_j^*}, g_{t_j^*}, x_{t_j^*})$ as above, where $t_j^* \in [t_j, t_j']$. Then by continuity, we obtain that there exists j_0 and $\tau_1 \leq \tau_0$ such that $\lambda_1(L_{g_j(\tau_1)}) > 0$ for all choices of $t_j \in [t_j, t_j']$ for all $j > j_0$. This means that there exists a psc-concordance between the original metrics g_0 and g_1 with $\Lambda \geq 0$.

This completes the case $\iota_0 = 0$, and the proof of Theorem 2.9. $\square$

11. Mean curvature and zero conformal class

11.1. **The setting.** In this section, we analyze a construction of gluing of two Riemannian manifolds with boundary along an embedded interval equipped with ε-standard metric. Let $(W_1, \hat{g}_1)$, $(W_2, \hat{g}_2)$ be two manifolds with the boundaries $(\partial W_1, \partial\hat{g}_1)$, $(\partial W_1, \partial\hat{g}_1)$ respectively. We have to consider the case when the manifold $(W_1, \hat{g}_1)$ is not necessarily compact. In that case, we have to assume that the scalar-flat problem on $(W_1, \hat{g}_1)$ makes sense and elliptic. This is guaranteed by the property that $(W_1, \hat{g}_1)$ has bounded $\mathbf{K}_q$-geometry. Nevertheless, we analyze this in two steps: first we assume

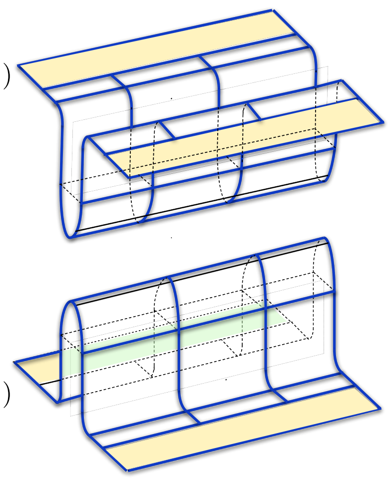

Figure 10. The manifolds $(W_1, \hat{g}_1)$ and $(W_2, \hat{g}_2)$ ready to be glued

that all manifolds are compact, and we analyze a relevant non-compact case in the next section. We assume that the manifolds $(W_1, \hat{g}_1)$, $(W_2, \hat{g}_2)$ are equipped with isometric embeddings

$$\begin{aligned} \iota_1 &: (D^{n-1}\times I, g_{\mathrm{torp}}^{(n-1)}(\varepsilon) + dt^2) \hookrightarrow (W_1, \hat{g}_1), \\ \iota_2 &: (D^{n-1}\times I, g_{\mathrm{torp}}^{(n-1)}(\varepsilon) + dt^2) \hookrightarrow (W_2, \hat{g}_2), \end{aligned} \tag{11.1}$$

such that

$$\begin{aligned} (D^{n-1}\times I, g_{\mathrm{torp}}^{(n-1)}(\varepsilon) + dt^2) \cap (\partial W_1, \hat{g}_1) &= (D^{n-1}\times \{0,1\}, g_{\mathrm{torp}}^{(n-1)}(\varepsilon)), \\ (D^{n-1}\times I, g_{\mathrm{torp}}^{(n-1)}(\varepsilon) + dt^2) \cap (\partial W_2, \hat{g}_2) &= (D^{n-1}\times \{0,1\}, g_{\mathrm{torp}}^{(n-1)}(\varepsilon)). \end{aligned} \tag{11.2}$$

Recall that the torpedo metric gives a decomposition (see Fig. 10):

$$\begin{aligned} (D^{n-1}\times I, g_{\mathrm{torp}}^{(n-1)}(\varepsilon) + dt^2) &= (S^{n-2}(\varepsilon)\times [0,a_0]\times I, g_o^{(n-2)} + ds^2 + dt^2) \\ &\cup (S^{n-2}\times [a_0,a_1]\times I, \bar{g}_o + dt^2) \cup (S^{n-1}_+(\varepsilon)\times I, g_o^{(n-1)} + dt^2). \end{aligned} \tag{11.3}$$

Here $(S^{n-2}(\varepsilon)\times[0,a_0], g_o^{(n-2)} + ds^2)$ is the standard cylinder, $(S^{n-1}_+(\varepsilon), g_o^{(n-1)})$ is a standard hemisphere of radius ε, and $(S^{n-2}(\varepsilon)\times[a_0,a_1], \bar{g}_o)$ is a transition region between the standard pieces.

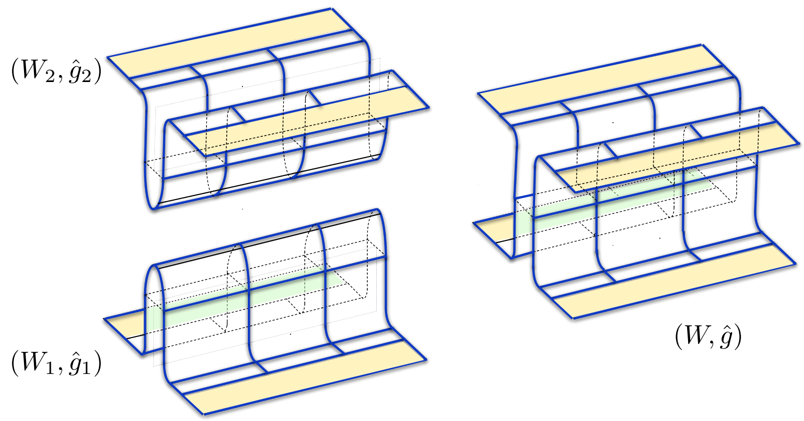

FIGURE 11. The manifold $(W, \hat{g})$

($\diamond$) Assumption. We assume that the decompositions (11.1), (11.2) and (11.3) are chosen the same for both manifolds $(W_1, \hat{g}_1)$ and $(W_2, \hat{g}_2)$.

Under the assumption ($\diamond$), we can glue the manifolds $(W_1, \hat{g}_1)$ and $(W_2, \hat{g}_2)$ by chopping off the cylinders of hemispheres together with the transition regions:

$$\begin{aligned} W_1' \;&:=\; W_1 \setminus \left((S^{n-2} \times [a_0, a_1] \times I) \cup (S^{n-1}_+(\varepsilon) \times I) \right) \\ W_2' \;&:=\; W_2 \setminus \left((S^{n-2} \times [a_0, a_1] \times I) \cup (S^{n-1}_+(\varepsilon) \times I) \right) \end{aligned} \tag{11.4}$$

and then gluing W_1' and W_2' together by identifying the cylindrical parts

$$\begin{aligned} U_1 &:= (S^{n-2}(\varepsilon) \times [0, a_0] \times I, g_o^{(n-2)} + ds^2 + dt^2) \subset W_1' \\ U_2 &:= (S^{n-2}(\varepsilon) \times [0, a_0] \times I, g_o^{(n-2)} + ds^2 + dt^2) \subset W_2' \end{aligned} \tag{11.5}$$

by the formula $(x, s, t) \mapsto (x, a_0 - s, t)$, see Fig. 14. We denote the resulting manifold $(W, \hat{g})$.

11.2. **Main result.** We need the following technical result.

Proposition 11.1. *Let $(W_1, \hat{g}_1)$, $(W_2, \hat{g}_2)$ be n-dimensional compact manifolds with boundaries $(\partial W_1, \partial \hat{g}_1)$, $(\partial W_1, \partial \hat{g}_1)$ equipped with the isometrics embeddings* (11.1) *satifying the Assumption* ($\diamond$). *Let $(W, \hat{g})$ be the manifold given by gluing $(W_1, \hat{g}_1)$ and $(W_2, \hat{g}_2)$ according to the formulas* (11.4) *and* (11.5). *Assume that $\mu_1(L_{\hat{g}_1}) = 0$ and $\mu_1(L_{\hat{g}_2}) = c > 0$. Then $\mu_1(L_{\hat{g}}) > 0$.*

Proof. We notice that since $\mu_1(L_{\hat{g}_1}) = 0$ and $\mu_1(L_{\hat{g}_2}) = c > 0$, the corresponding Yamabe constants $Y_{[\hat{g}_1]}(W_1, \partial W_1) = 0$, and $Y_{[\hat{g}_2]}(W_2, \partial W_2) = c' > 0$ respectively. We choose smooth functions u_1 and u_2 on W_1 and W_2 respectively such that the metrics $\tilde{g}_1 = u_1^{\frac{4}{n-2}} \hat{g}_1$, $\tilde{g}_2 = u_2^{\frac{4}{n-2}} \hat{g}_2$ are scalar-flat, and

have the following mean curvatures:

$$
(11.6)\qquad
\begin{cases}
a_n\Delta_{\hat{g}_1}u_1 + R_{\hat{g}_1}u_1 \equiv 0 & \text{on } W_1,\\
h_{\tilde{g}_1} = u_1^{-\frac{n}{n-2}}(\tfrac{2}{n-2}\partial_{\hat{\nu}}u_1 + h_{\hat{g}_1}u_1) = 0 & \text{along } \partial W_1
\end{cases}
\qquad
\begin{cases}
a_n\Delta_{\hat{g}_2}u_2 + R_{\hat{g}_2}u_2 \equiv 0 & \text{on } W_2,\\
h_{\tilde{g}_2} = u_2^{-\frac{n}{n-2}}(\tfrac{2}{n-2}\partial_{\hat{\nu}}u_2 + h_{\hat{g}_2}u_2) = c' & \text{along } \partial W_2
\end{cases}
$$

We choose the following normalization for the functions u_1 and u_2

$$
(11.7)\qquad \int_{\partial W_1} u_1^{\frac{2(n-1)}{n-2}} d\sigma_{\partial\hat{g}_1} = 1, \quad \int_{\partial W_2} u_2^{\frac{2(n-1)}{n-2}} d\sigma_{\partial\hat{g}_2} = 1.
$$

Now we construct and analyze the metric $\tilde{g}_\delta$ conformal to the metric $\hat{g}$.

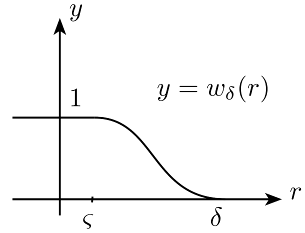

FIGURE 12. The cut-off function $w_\delta(r)$.

Step 1: We choose a cut-off function. Let δ be a small constant as above, $0<\delta<a_0$, and $\varsigma := \frac{1}{4}e^{-\frac{1}{\delta}}$. We denote by w_δ a smooth nonnegative function such that

(i) $\begin{cases} w_\delta(r)\equiv 1 & \text{on } [0,\varsigma],\\ w_\delta(r)\equiv 0 & \text{on } [\delta,\infty),\end{cases}$

(ii) $|r\dot{w}_\delta(r)| < \delta$ for $r\geq 0$,

(iii) $|r\ddot{w}_\delta(r)| < \delta$ for $r\geq 0$, see Fig. 12.

The function $w_\delta(r)$ was introduced by O. Kobayashi, [22], see also [1].

Step 2: We glue together the metrics $\tilde{g}_1$ and $\tilde{g}_2$. We consider the cylindrical parts U_1 and U_2 as above (11.5), and identify U_1 and U_2 via the map $(x,s,t)\mapsto(x,a_0-s,t)$ as above. The metrics $\hat{g}_1$ and $\hat{g}_2$ restricted on $U := U_1 = U_2$ coincide, and we denote by $\hat{g}$ the resulting metric on W as above. We also denote:

$$
\hat{g}_{12} := \hat{g}_1|_{U_1=U} = \hat{g}_2|_{U_2=U}.
$$

We introduce the coordinates (x,s,t) on the cylinder U, and we decompose the functions u_1, u_2 restricted to the cylinder U:

$$
(11.8)\qquad
\begin{aligned}
u_1(x,s,t) &= u_1^0(x,t) + su_1^1(x,t) + \tfrac{s^2}{2}f_1(x,s,t),\\
u_2(x,s,t) &= u_2^0(x,t) + su_2^1(x,t) + \tfrac{s^2}{2}f_2(x,s,t).
\end{aligned}
$$

Since the metrics $\tilde{g}_1$ and $\tilde{g}_2$ are scalar-flat, on the cylinder U, we have

$$\begin{aligned} &\alpha_n \Delta_{\hat{g}_{12}} u_1 + R_0 u_1 = R_{\tilde{g}_1} u_1^{\frac{n+2}{n-2}} = 0, \\ &\alpha_n \Delta_{\hat{g}_{12}} u_2 + R_0 u_2 = R_{\tilde{g}_1} u_2^{\frac{n+2}{n-2}} = 0, \end{aligned} \tag{11.9}$$

where $R_0 = R_{\hat{g}_{12}}$ is a positive constant. Then we define the metric $\tilde{g}_\delta$ conformal to $\hat{g}$ on W as follows. First we let

$$u_\delta(x,s,t) = (1-w_\delta(s)) \cdot u_1(x,s,t) + w_\delta(s) \cdot u_2(x,s,t). \tag{11.10}$$

Then we let $\tilde{g}_\delta = u_\delta^{\frac{4}{n-2}} \hat{g}$, i.e., we obtain:

$$\tilde{g}_\delta = \begin{cases} u_1^{\frac{4}{n-2}} \hat{g}_1 & \text{on } W_1 \setminus U_1, \\ ((1-w_\delta)\cdot u_1 + w_\delta \cdot u_2)^{\frac{4}{n-2}} \hat{g}_{12} & \text{on } S^{n-1} \times [0,a_0] \times I \\ u_2^{\frac{4}{n-2}} \hat{g}_2 & \text{on } W_2 \setminus U_2, \end{cases} \tag{11.11}$$

We emphasize that the metric $\tilde{g}_\delta$ depends on δ via the cut-off function w_δ. For the future use, we notice that the normalization (11.7) implies the inequality

$$\int_{\partial W} u_\delta^{\frac{2(n-1)}{n-2}} d\sigma_{\partial \hat{g}} \quad \leq \quad \int_{\partial W_1} u_1^{\frac{2(n-1)}{n-2}} d\sigma_{\partial \hat{g}_1} + \int_{\partial W_2} u_2^{\frac{2(n-1)}{n-2}} d\sigma_{\partial \hat{g}_2} \leq 2. \tag{11.12}$$

Step 3: We compute the scalar curvature of the metric $\tilde{g}_\delta$. First we notice that in the above coordinates on the cylinder U, we have $\Delta_{\hat{g}_{12}} = \Delta_{\hat{g}^o} + \frac{d^2}{ds^2}$, where $\hat{g}^o = g_{\mathrm{torp}}^{(n-1)} + dt^2$. We have

$$\begin{aligned} u_\delta &= (1-w_\delta)u_1 + w_\delta u_2 \\ &= (1-w_\delta)u_1^0 + w_\delta u_2^0) + s((1-w_\delta)u_1^1 + w_\delta u_2^1) + \tfrac{s^2}{2}((1-w_\delta)f_1 + w_\delta f_2). \end{aligned}$$

We recall that u_i^0 and u_i^1 do not depend on s. We have:

$$\begin{aligned} \Delta_{\hat{g}_{12}}\left((1-w_\delta)u_1^0 + w_\delta u_2^0)\right) &= (1-w_\delta)\Delta_{\hat{g}^o} u_1^0 + w_\delta \Delta_{\hat{g}^o} u_2^0 - w_\delta''(u_1^0 - u_2^0), \\ \Delta_{\hat{g}_{12}}\left(s((1-w_\delta)u_1^1 + w_\delta u_2^1))\right) &= s((1-w_\delta)\Delta_{\hat{g}^o} u_1^1 + w_\delta \Delta_{\hat{g}^o} u_2^1) \\ &\quad -2sw_\delta'(u_1^1 - u_2^1) - sw_\delta''(u_1^1 - u_2^1), \\ \Delta_{\hat{g}_{12}}(\tfrac{s^2}{2}((1-w_\delta)f_1 + w_\delta f_2))) &= \tfrac{s^2}{2}((1-w_\delta)\Delta_{\hat{g}^o} f_1 + w_\delta \Delta_{\hat{g}^o} f_2) \\ &\quad +(\tfrac{s^2}{2}((1-w_\delta)f_1 + w_\delta f_2)))''. \end{aligned}$$

Here $'$ and $''$ denote the partial derivatives with respect to s. We have:

$$\left(\tfrac{s^2}{2}((1-w_\delta)f_1+w_\delta f_2)\right)''=\tfrac{s^2}{2}((1-w_\delta)f_1+w_\delta f_2))''+s((1-w_\delta)f_1+w_\delta f_2))'$$
$$+((1-w_\delta)f_1+w_\delta f_2))$$

We compute:

$$\begin{aligned}\tfrac{s^2}{2}((1-w_\delta)f_1+w_\delta f_2))'' &= \tfrac{s^2}{2}((1-w_\delta)f_1''+w_\delta f_2'')-s^2w_\delta'(f_1'-f_2')-\tfrac{s^2}{2}w_\delta''(f_1-f_2),\\ s((1-w_\delta)f_1+w_\delta f_2))' &= s((1-w_\delta)f_1'+w_\delta f_2'))-sw_\delta'(f_1-f_2)\end{aligned}$$

Then we have:

$$\begin{aligned}\Delta_{\hat g_{12}}u_\delta &= (1-w_\delta)\left[\Delta_{\hat g^o}u_1^0+s\Delta_{\hat g^o}u_1^1+(\tfrac{s^2}{2}\Delta_{\hat g^o}f_1+f_1+sf_1'+\tfrac{s^2}{2}f_1'')\right]\\ &\quad +w_\delta\left[\Delta_{\hat g^o}u_2^0+s\Delta_{\hat g^o}u_2^1+(\tfrac{s^2}{2}\Delta_{\hat g^o}f_2+f_2+sf_2'+\tfrac{s^2}{2}f_2'')\right]\\ &\quad -w_\delta''(u_1^0-u_2^0)-2sw_\delta'(u_1^1-u_2^1)-sw_\delta''(u_1^1-u_2^1)-sw_\delta'(f_1-f_2)\\ &\quad -s^2w_\delta'(f_1'-f_2')-\tfrac{s^2}{2}w_\delta''(f_1-f_2)\\ &= (1-w_\delta)\Delta_{\hat g_{12}}u_1+w_\delta\Delta_{\hat g_{12}}u_2-w_\delta''(u_1^0-u_2^0)\\ &\quad -s\left(2w_\delta'(u_1^1-u_2^1)+w_\delta''(u_1^1-u_2^1)+w_\delta'(f_1-f_2)\right)\\ &\quad -\tfrac{s^2}{2}\left(2w_\delta'(f_1'-f_2')+w_\delta''(f_1-f_2)\right)\end{aligned}\tag{11.13}$$

We notice that the term $w_\delta''(u_1^0-u_2^0)$ vanishes: indeed, since $w_\delta''(0)=0$, we have:

$$w_\delta''(u_1^0(x,t)-u_2^0(x,t))=w_\delta''(0)(u_1(x,t,0)-u_2(x,t,0))=0.$$

Recall that the scalar curvature $R_{\hat g_{12}}=R_0>0$ is a constant on the cylinder U. Thus we have:

$$\begin{aligned}\alpha_n\Delta_{\hat g_{12}}u_\delta+R_0u_\delta &= \alpha_n\left((1-w_\delta)\Delta_{\hat g_{12}}u_1+w_\delta\Delta_{\hat g_{12}}u_2\right)+R_0((1-w_\delta)u_1+w_\delta u_2)\\ &\quad -s\left(2w_\delta'(u_1^1-u_2^1)+w_\delta''(u_1^1-u_2^1)+w_\delta'(f_1-f_2)\right)\\ &\quad -\tfrac{s^2}{2}\left(2w_\delta'(f_1'-f_2')+w_\delta''(f_1-f_2)\right)\end{aligned}$$

Then we continue:

$$\begin{aligned}\alpha_n\Delta_{\hat{g}_{12}}u_\delta + R_0 u_\delta \;=\; & \underbrace{(1-w_\delta)(\alpha_n\Delta_{\hat{g}_{12}}u_1 + R_0u_1) + w_\delta(\alpha_n\Delta_{\hat{g}_{12}}u_2 + R_0u_2)}_{=0} \\ & -s\left(2w_\delta'(u_1^1-u_2^1) + w_\delta''(u_1^1-u_2^1) + w_\delta'(f_1-f_2)\right) \\ & -\tfrac{s^2}{2}\left(2w_\delta'(f_1'-f_2') + w_\delta''(f_1-f_2)\right).\end{aligned}$$

Now we would like to get an estimate on the scalar curvature

$$R_{\tilde{g}_\delta} = u_\delta^{-\frac{n+2}{n-2}}(\alpha_n\Delta_{\hat{g}_{12}}u_\delta + R_0u_\delta).$$

Lemma 11.2. *There exists a constant $B>0$ which does not depend on δ, such that*

$$\begin{aligned}2|sw_\delta'|\cdot|u_1^1-u_2^1| + |sw_\delta''|\cdot|u_1^1-u_2^1| + |sw_\delta'|\cdot|f_1-f_2| + \\ s|sw_\delta'|\cdot|f_1'-f_2'| + \tfrac{s}{2}|s\theta_\delta''|\cdot|f_1-f_2| \quad < \delta\cdot B.\end{aligned}$$

Proof. Recall that

$$|sw_\delta'(s)| < \delta \quad \text{and} \quad |sw_\delta''(s)| < \delta$$

for all $s\in[0,\delta]$, $\delta < a_0$. Then the properties (ii^o) and (iii^o) of w_δ imply:

$$\begin{aligned}& 2|sw_\delta'|\cdot|u_1^1-u_2^1| + |sw_\delta''|\cdot|u_1^1-u_2^1| + |sw_\delta'|\cdot|f_1-f_2| \\ & \quad + s|sw_\delta'|\cdot|f_1'-f_2'| + \tfrac{s}{2}|sw_\delta''|\cdot|f_1-f_2| \\ & < \delta(2|u_1^1-u_2^1| + |u_1^1-u_2^1| + |f_1-f_2| + a_0|f_1'-f_2'| + \tfrac{a_0}{2}|f_1-f_2|) < \delta\cdot B,\end{aligned}$$

where $B = \max\{1, (2|u_1^1-u_2^1| + |u_1^1-u_2^1| + |f_1-f_2| + a_0|f_1'-f_2'| + \frac{a_0}{2}|f_1-f_2|)\}$, and the maximum is taken over U. □

Lemma 11.3. *There exist $D_0>0$, $A_0>0$, such that $D_0 \geq u_\delta \geq A_0$ on U, where A_0 and D_0 do not depend on δ.*

Proof. Since u_1 and u_2 are smooth positive functions and U is compact, there exist $D_0>0$, $A_0>0$ such that $u_1,u_2\geq A_0$, and $u_1,u_2\leq D_0$ on U. Then we have:

$$u_\delta = (1-w_\delta)u_1 + w_\delta u_2 \geq (1-\theta_\delta)A_0 + w_\delta A_0 = A_0$$

since $0\leq w_\delta\leq 1$. Similarly, $u_\delta \leq D_0$. □

We denote by $A_1 := A_0^{-\frac{n+2}{n-2}}$. We keep in mind that $R_{\tilde{g}_\delta}\equiv 0$ outside of U by construction, and we obtain on U:

$$|R_{\tilde{g}_\delta}| \;=\; u_\delta^{-\frac{n+2}{n-2}}|\alpha_n\Delta_{\hat{g}_{12}}u_\delta + R_0u_\delta| < A_1\cdot B\cdot\delta \tag{11.14}$$

Since $\varsigma < \delta < a_0$, the inequality (11.14) gives an estimate:

Lemma 11.4. *There is an inequality*

$$\begin{cases} R_{\tilde{g}_\delta} & = & 0 & \textit{outside of } U \\ |R_{\tilde{g}_\delta}| & < & A_1 \cdot B \cdot \delta & \textit{on } U \end{cases} \tag{11.15}$$

where the constants A_1 and B do not depend on δ.

Remark. We emphasize that the constants A_1 and B are all determined by the functions u_1, u_2 restricted to the cylinder U.

Step 4: We compute the mean curvature of the metric $\tilde{g}_\delta$. Let (x, s, t) be the coordinates on the cylinder U. By construction, the normal vector field $\hat{\nu}$ is perpendicular to the s-coordinate. By definition of the functions u_1, u_2, (11.6), (11.8), we have:

$$\tfrac{2}{n-2}\partial_{\hat{\nu}} u_1 = -h_{\hat{g}_{12}} u_1,$$

$$\tfrac{2}{n-2}\partial_{\hat{\nu}} u_2 = -h_{\hat{g}_{12}} u_2 + c' u_2^{\frac{n}{n-2}}$$

Thus we have:

$$\begin{array}{rcl} \tfrac{2}{n-2}\partial_{\hat{\nu}} u_\delta & = & \tfrac{2}{n-2}\partial_{\hat{\nu}}((1-w_\delta)u_1 + w_\delta u_2) \\[2ex] & = & (1-w_\delta)\tfrac{2}{n-2}\partial_{\hat{\nu}} u_1 + w_\delta \tfrac{2}{n-2}\partial_{\hat{\nu}} u_2 \\[2ex] & = & (1-w_\delta)(-h_{\hat{g}_{12}} u_1) + w_\delta(-h_{\hat{g}_{12}} u_2 + c' u_2^{\frac{n}{n-2}}) \\[2ex] & = & -h_{\hat{g}_{12}}\left((1-w_\delta)u_1 + w_\delta u_2\right) + \theta_\delta c' u_2^{\frac{n}{n-2}} \\[2ex] & = & -h_{\hat{g}_{12}} u_\delta + w_\delta c' u_2^{\frac{n}{n-2}}. \end{array}$$

Then we have on U:

$$\begin{array}{rcl} h_{\tilde{g}_\delta} & = & u_\delta^{-\frac{n}{n-2}}\left(\tfrac{2}{n-2}\partial_{\hat{\nu}} u_\delta + h_{\hat{g}_{12}} u_\delta\right) \\[2ex] & = & u_\delta^{-\frac{n}{n-2}}\left(-h_{\hat{g}_{12}} u_\delta + w_\delta c' u_2^{\frac{n}{n-2}} + h_{\hat{g}_{12}} u_\delta\right) \\[2ex] & = & u_\delta^{-\frac{n}{n-2}} c' \cdot w_\delta \cdot u_2^{\frac{n}{n-2}} \end{array}$$

We recall that $u_\delta = w_\delta u_2 = u_2$ on $W_2 \setminus U$. Thus on $W_2 \setminus U$, we obtain:

$$h_{\tilde{g}_\delta} \;=\; u_\delta^{-\frac{n}{n-2}} \cdot c' \cdot u_2^{\frac{n}{n-2}} \geq u_2^{-\frac{n}{n-2}} \cdot c' \cdot u_2^{-\frac{n}{n-2}} = c'.$$

We summarize the the estimates:

Lemma 11.5.

$$
\begin{cases} h_{\tilde{g}_\delta} & = & 0 & along\ \partial W_1 \setminus \partial U \\ h_{\tilde{g}_\delta} & > & 0 & along\ \partial U \\ h_{\tilde{g}_\delta} & > & c' > 0 & along\ \partial W_2 \setminus \partial U \end{cases} \tag{11.16}
$$

Step 5: We assume that $\mu_1(L_{\tilde{g}_\delta}) \le 0$. First we recall μ_1 could be defined through the Rayleigh quotient

$$
\mu_1(L_{\tilde{g}_\delta}) = \inf_{f\in C^\infty_+} \frac{E_{\tilde{g}_\delta}(f)}{\int_{\partial W} f^2 d\sigma_{\partial\tilde{g}_\delta}}, \tag{11.17}
$$

where

$$
E_{\tilde{g}_\delta}(f) := \int_W (a_n|\nabla_{\tilde{g}_\delta} f|^2 + R_{\tilde{g}_\delta} f^2) d\sigma_{\tilde{g}_\delta} + 2(n-1)\int_{\partial W} h_{\tilde{g}_\delta} f^2 d\sigma_{\partial\tilde{g}_\delta}
$$

We assume that $\int_{\partial W} f^2 d\sigma_{\partial\tilde{g}_\delta} = 1$ and consider the numerator $E_{\tilde{g}_\delta}(f)$. The term $a_n|\nabla_{\tilde{g}_\delta} f|^2$ is positive, so we would like to compare the terms

$$
\int_W R_{\tilde{g}_\delta} f^2 d\sigma_{\tilde{g}_\delta} \quad \text{and} \quad 2(n-1)\int_{\partial W} h_{\tilde{g}_\delta} f^2 d\sigma_{\partial\tilde{g}_\delta},
$$

where the first one could be negative, and the second term is certainly positive.

Since $d\sigma_{\tilde{g}_\delta} = u_\delta^{\frac{2n}{n-2}} d\sigma_{\hat{g}}$, and $D_0 \ge u_\delta \ge A_0$, we obtain that

$$
\mathrm{Vol}_{\tilde{g}_\delta}(U) = A_0^{\frac{2n}{n-2}} \cdot \mathrm{Vol}_{\hat{g}}(U),
$$

where $A_0 > 0$ is from Lemma 11.3, and it does not depend on δ. We denote $A_4 = A_0^{\frac{2n}{n-2}}$. Keeping in mind that $R_{\tilde{g}_\delta} = 0$ outside of U, we have:

$$
\begin{aligned} \left| \int_W R_{\tilde{g}_\delta} d\sigma_{\tilde{g}_\delta} \right| &< A_1 \cdot B \cdot \delta \int_U d\sigma_{\tilde{g}_\delta} \\ &= A_1 \cdot B \cdot A_4 \cdot \delta \cdot \mathrm{Vol}_{\hat{g}}(U) = A_{**} \cdot \delta \end{aligned} \tag{11.18}
$$

where $A_{**} = A_1 \cdot B \cdot A_4 \cdot \mathrm{Vol}_{\hat{g}}(U)$. Similarly, we notice that $\mathrm{Vol}_{\partial\tilde{g}_\delta}(\partial W_2') = A_5 \cdot \mathrm{Vol}_{\partial\hat{g}}(\partial W_2')$, where $A_5 > 0$ does not depend on δ. Keeping in mind that $h_{\tilde{g}_\delta}$ vanishes outside of $\partial W_2'$, we have:

$$
\begin{aligned} 2(n-1)\int_{\partial W} h_{\tilde{g}_\delta} d\sigma_{\partial\tilde{g}_\delta} &> 2(n-1)c' \cdot \int_{\partial W_2'} d\sigma_{\partial\tilde{g}_\delta} \\ &= 2(n-1)c' \cdot \int_{\partial W_2'} \sigma^{\frac{2(n-1)}{n-2}} d\sigma_{\partial\hat{g}_2} = D_{**}, \end{aligned} \tag{11.19}
$$

where $D_{**} = 2(n-1) \cdot c' \cdot \mathrm{Vol}_{\partial\hat{g}_2}(\partial W_2')$.

We recall that $\tilde{g}_\delta \in [\hat{g}]$. Thus the signs of the eigenvalues $\mu_1(L_{\tilde{g}_\delta})$ and $\mu_1(L_{\hat{g}})$ are the same. Thus $\mu_1(L_{\tilde{g}_\delta}) \le 0$ and $\mu_1(L_{\hat{g}}) \le 0$.

Let ζ be a solution of the corresponding scalar-flat Yamabe problem on $(W,\hat{g})$, i.e., the metric $\breve{g}=\zeta^{\frac{4}{n-2}}\hat{g}$ is such that

$$\left\{\begin{array}{ll} R_{\breve{g}}\equiv 0 & \text{on } W \\ h_{\breve{g}}=Y^{\flat}_{[\hat{g}]}(W,\partial W)\leq 0 & \text{along } \partial W. \end{array}\right. \tag{11.20}$$

Here $Y^{\flat}_{[\hat{g}]}(W,\partial W)$ is the corresponding scalar-flat Yamabe constant. It follows from [16] that such a solution ζ always exists for nonpositive conformal classes. We use the standard normalization of the function ζ:

$$\int_{\partial W}\zeta^{\frac{2(n-1)}{n-2}}d\sigma_{\partial\hat{g}}=1. \tag{11.21}$$

Since the metric $\tilde{g}_\delta$ is in the same conformal class $[\hat{g}]$ for any $\delta>0$, there exists a unique function v_δ such that $\breve{g}=v_\delta^{\frac{4}{n-2}}\tilde{g}_\delta$. Recall that $\tilde{g}_\delta=u_\delta^{\frac{4}{n-2}}\breve{g}$, i.e., $\breve{g}=u_\delta^{-\frac{4}{n-2}}\tilde{g}_\delta$. We obtain:

$$\breve{g}=\zeta^{\frac{4}{n-2}}\hat{g}=\zeta^{\frac{4}{n-2}}u_\delta^{-\frac{4}{n-2}}\tilde{g}_\delta=v_\delta^{\frac{4}{n-2}}\tilde{g}_\delta.$$

Thus $v_\delta=\zeta u_\delta^{-1}$.

Lemma 11.6. *The following estimate holds:*

$$\left(\int_{\partial W}v_\delta^2 d\sigma_{\partial\tilde{g}_\delta}\right)^{\frac{1}{2}}\leq 2$$

Proof. Since $v_\delta=\zeta u_\delta^{-1}$, and $d\sigma_{\partial\tilde{g}_\delta}=u_\delta^{\frac{2(n-1)}{n-2}}d\sigma_{\partial\hat{g}}$, we have

$$\begin{aligned}\int_{\partial W}v_\delta^2 d\sigma_{\partial\tilde{g}_\delta} &= \int_{\partial W}\zeta^2u_\delta^{-2}d\sigma_{\partial\tilde{g}_\delta}=\int_{\partial W}\zeta^2u_\delta^{-2}u_\delta^{\frac{2(n-1)}{n-2}}d\sigma_{\partial\hat{g}}\\ &= \int_{\partial W}\zeta^2u_\delta^{\frac{2}{n-2}}d\sigma_{\partial\hat{g}}\end{aligned}$$

Now we would like to use the Hölder inequality corresponding to the parameters $\frac{1}{2}=\frac{n-2}{2(n-1)}+\frac{1}{2(n-1)}$, so $p=\frac{2(n-1)}{n-2}$, $q=2(n-1)$. Then we have:

$$\begin{aligned}\left(\int_{\partial W}\zeta^2u_\delta^{\frac{2}{n-2}}d\sigma_{\partial\hat{g}}\right)^{\frac{1}{2}} &\leq \left(\int_{\partial W}\zeta^p d\sigma_{\partial\hat{g}}\right)^{\frac{1}{p}}\cdot\left(\int_{\partial W}u_\delta^{\frac{q}{n-2}}d\sigma_{\partial\hat{g}}\right)^{\frac{1}{q}}\\ &= \left(\int_{\partial W}\zeta^{\frac{2(n-1)}{n-2}}d\sigma_{\partial\hat{g}}\right)^{\frac{n-2}{2(n-1)}}\cdot\left(\int_{\partial W}u_\delta^{\frac{2(n-1)}{n-2}}d\sigma_{\partial\hat{g}}\right)^{\frac{1}{2(n-1)}}\\ &\leq 1\cdot 2^{\frac{1}{2(n-1)}}<2.\end{aligned}$$

The last inequality follows from the normalization (11.21) and the bound (11.12). $\square$

Recall that $D_0 \geq u_\delta \geq A_0$ on U, for some constants $A_0, D_0 > 0$ which do not depend on δ. Hence we have:

$$A_0^{-1} \geq u_\delta^{-1} \geq D_0^{-1}$$

The function $\zeta > 0$ is uniquely determined by the metric $\hat{g}$, in particular, it does not depend on δ. In particular, there exist A_6 and D_6 such that

$$A_6 \geq \zeta \geq D_6$$

on U. We obtain the inequality

$$A_7 \geq v_\delta \geq D_7$$

on U, where $A_7, D_7 > 0$ do not depend on δ.

Now, since g_δ is in the same conformal class as $\hat{g}$ and $\check{g}$. Thus we must have that $E_{\tilde{g}_\delta}(v_\delta) \leq 0$ for all $\delta > 0$. We have:

$$\begin{aligned} E_{\tilde{g}_\delta}(v_\delta) &= \int_W (a_n |\nabla_{\tilde{g}_\delta} v_\delta|^2 + R_{\tilde{g}_\delta} v_\delta^2) d\sigma_{\tilde{g}_\delta} + 2(n-1) \int_{\partial W} h_{\tilde{g}_\delta} v_\delta^2 d\sigma_{\partial \tilde{g}_\delta} \\ &> -\left| \int_W R_{\tilde{g}_\delta} v_\delta^2 d\sigma_{\tilde{g}_\delta} \right| + 2(n-1) \int_{\partial W} h_{\tilde{g}_\delta} v_\delta^2 d\sigma_{\partial \tilde{g}_\delta} \\ &= -\int_W |R_{\tilde{g}_\delta}|\, v_\delta^2 d\sigma_{\tilde{g}_\delta} + 2(n-1) \int_{\partial W} h_{\tilde{g}_\delta} v_\delta^2 d\sigma_{\partial \tilde{g}_\delta} \end{aligned} \tag{11.22}$$

Then we have:

$$\begin{aligned} -\int_W |R_{\tilde{g}_\delta}|\, v_\delta^2 d\sigma_{\tilde{g}_\delta} &\geq -D_7^2 \int_W |R_{\tilde{g}_\delta}|\, d\sigma_{\tilde{g}_\delta} \\ &> -D_7^2 A_{**} \cdot \delta = -A_* \cdot \delta. \end{aligned} \tag{11.23}$$

where $A_* = D_7^2 \cdot A_{**}$. We emphasize that we could replaced $\mathrm{Vol}_{\hat{g}}(W)$ by $\mathrm{Vol}_{\hat{g}}(U)$ in (11.23) since the scalar curvature $R_{\tilde{g}_\delta}$ vanishes outside of U. Recall the notation:

$$W_2' = W_2 \setminus \left((S^{n-2} \times [a_0, a_1] \times I) \cup (S_+^{n-1}(\varepsilon) \times I) \right),$$

and consider its boundary $\partial W_2' = \partial W_2 \setminus S_+^{n-1}(\varepsilon) \times \{0,1\}$. Since $h_{\tilde{g}_\delta}$ vanishes outside of $\partial W_2'$, we have that $\int_{\partial W} h_{\tilde{g}_\delta} v_\delta^2 d\sigma_{\partial \tilde{g}_\delta} = \int_{\partial W_2'} h_{\tilde{g}_\delta} v_\delta^2 d\sigma_{\partial \tilde{g}_\delta}$. We obtain:

$$2(n-1) \int_{\partial W} h_{\tilde{g}_\delta} v_\delta^2 d\sigma_{\partial \tilde{g}_\delta} \quad \geq \quad 2(n-1) A_7^2 \cdot D_{**} \cdot \int_{\partial W} h_{\tilde{g}_\delta} d\sigma_{\partial \tilde{g}_\delta} \geq D_*$$

where $D_* = 2(n-1) A_7^2 \cdot D_{**}$. We continue the inequality (11.22):

$$\begin{aligned} E_{\tilde{g}_\delta}(v_\delta) &> -\int_W |R_{\tilde{g}_\delta}|\, v_\delta^2 d\sigma_{\tilde{g}_\delta} + 2(n-1) \int_{\partial W} h_{\tilde{g}_\delta} v_\delta^2 d\sigma_{\partial \tilde{g}_\delta} \\ &\geq -A_* \delta + D_* \end{aligned} \tag{11.24}$$

Here $A_* > 0$ and $D_* > 0$ do not depend on δ. Clearly the linear form $-A_* \delta + D_* > 0$ for small enough $\delta > 0$. On the other hand, by assumption, $E_{\tilde{g}_\delta}(v_\delta) \leq 0$. This provides a contradiction to the assumption that $\mu_1(L_{\bar{g}}) \leq 0$ and completes the proof of Proposition 11.1. $\square$

11.3. **Gluing metrics of positive scalar curvature.** Here we analyze a gluing procedure in the case when the manifolds $(W_1,\hat{g}_1)$, $(W_2,\hat{g}_2)$ have positive scalar curvature. Namely, we prove the following:

Proposition 11.7. *Let $(W_1,\hat{g}_1)$, $(W_2,\hat{g}_2)$ be n-dimensional compact manifolds with boundaries $(\partial W_1,\partial\hat{g}_1)$, $(\partial W_2,\partial\hat{g}_2)$ equipped with the isometrics embeddings* (11.1) *satifying the Assumption* ($\diamond$)*. Let $(W,\hat{g})$ be the manifold given by gluing $(W_1,\hat{g}_1)$ and $(W_2,\hat{g}_2)$ according to the formulas* (11.4) *and* (11.5)*. Assume that $\lambda_1(L_{\hat{g}_1})>0$ and $R_{\hat{g}_2}\geq R_0$ and $h_{\hat{g}_2}=0$ along ∂W_2. Denote by u_1 the corresponding eigenfunction and by $\tilde{g}_1:=u_1^{\frac{4}{n-2}}\hat{g}_1$ the conformal metric on W_1, and by $\tilde{\iota}_1$ and $\hat{\iota}_2$ the injectivity radii of the metric $\tilde{g}_1$ and $\hat{g}_2$ respectively. Let u_1 be an eigenfunction of the minimal-boundary problem, i.e.,*

$$\begin{cases} a_n\Delta_{\hat{g}_1}u_1+R_{\hat{g}_1}u_1 \equiv 0 & \text{on } W_1,\\ \frac{2}{n-2}\partial_{\hat{\nu}}u_1+h_{\hat{g}_1}u_1 = 0 & \text{along } \partial W_1\end{cases}$$

Then there exists a function u on W such that

(i) *$u=u_1$ on $W_1\setminus U$ and $u=1$ on $W_2\setminus U$;*

(ii) *the metric $\tilde{g}=u^{\frac{4}{n-2}}\hat{g}$ has positive scalar curvature and $h_{\tilde{g}}=0$ along ∂W.*

(iii) *the injectivity radius $\tilde{\iota}$ of the metric $\tilde{g}$ is bounded below by* $\min\{\tilde{\iota}_1,\hat{\iota}_2\}$.

Proof. We use the same cut-off function w_δ as above. Recall that U is the cylindrical part

$$U=(S^{n-2}(\varepsilon)\times[0,a_0]\times I, g_o^{(n-2)}+ds^2+dt^2)$$

which we identify within W_1 and W_2 as above. We assume $\delta<a_0$ and define the function u_δ on the cylinder U:

$$u_\delta(x,s,t)=(1-w_\delta(s))\cdot u_1(x,s,t)+w_\delta(s)\cdot 1,$$

so that $u_\delta=u_1$ on $W_1\setminus U$ and $u=1$ on $W_2\setminus U$. We decompose the function u_1 on the cylinder U as above:

$$u_1(x,s,t)=u_1^0(x,t)+su_1^1(x,t)+\tfrac{s^2}{2}f_1(x,s,t).$$

Again, we let $\tilde{g}_\delta=u_\delta^{\frac{4}{n-2}}\hat{g}$, i.e., we obtain:

$$\tilde{g}_\delta=\begin{cases} u_1^{\frac{4}{n-2}}\hat{g}_1 & \text{on } W_1\setminus U_1,\\ ((1-w_\delta)\cdot u_1+w_\delta)^{\frac{4}{n-2}}\hat{g}_{12} & \text{on } S^{n-1}\times[0,a_0]\times I\\ \hat{g}_2 & \text{on } W_2\setminus U_2,\end{cases} \tag{11.25}$$

We denote by $\hat{g}_{12}$ the metric $\hat{g}_1$ or $\hat{g}_2$ restricted to the cylinder U, and $\lambda_1:=\lambda_1(L_{\hat{g}_1})$. Also recall that $R_{\hat{g}_{1,2}}=R_0$. By definition, on the cylinder U, we have

$$L_{\hat{g}_{1,2}}u_1=a_n\Delta_{\hat{g}_{1,2}}u_1+R_0u_1=\lambda_1u_1, \quad \partial_{\hat{\nu}_1}=0.$$

Then we use (11.13) to obtain:

$$\begin{aligned} L_{\hat{g}_{1,2}} u_\delta &= L_{\hat{g}_{1,2}}((1-w_\delta)u_1 + w_\delta) \\ &= (1-w_\delta)\lambda_1 u_1 + w_\delta R_0 - s\left(2w'_\delta u_1^1 + w''_\delta u_1^1 + w'_\delta f_1\right) - \tfrac{s^2}{2}\left(2w'_\delta f'_1 + w''_\delta f_1\right). \end{aligned} \tag{11.26}$$

Then we have the estimate:

$$\begin{aligned} \left|s\left(2w'_\delta u_1^1 + w''_\delta u_1^1 + w'_\delta f_1\right) - \tfrac{s^2}{2}\left(2w'_\delta f'_1 + w''_\delta f_1\right)\right| &\leq |sw'_\delta|(2|u_1^1| + |f_1| + 2s|f'_1|) \\ &\quad + |sw''_\delta|(|u_1^1| + \tfrac{s}{2}|f_1|) \ \leq \ \delta C_\bullet, \end{aligned} \tag{11.27}$$

where $C_\bullet = \max\{(2|u_1^1|+|f_1|+2s|f'_1|), (|u_1^1|+\frac{s}{2}|f_1|)\}$, and the maximum is taken over the cylindrical part U. Then we denote by $R_1 > 0$ the manimum of the function $(1-w_\delta)\lambda_1 u_1 + w_\delta R_0$ over the cylindrical part U. Then we find $\delta_0 > 0$ such that $R_1 > \delta_0 C_\bullet$. Let $u := u_{\delta_0}$. Then the condition (i) holds. We recall that $\partial_{\hat{\nu}} u_1 = 0$. Then it is easy to check that $\partial_{\hat{\nu}} u_\delta = 0$ for each $\delta > 0$, thus (ii) holds as well. It is easy to check that the condition (iii) holds as well. This proves Proposition 11.7. □

12. Surgery Lemma for concordances

The goal of this section is to prove Theorem 2.3. To make the constructions transparent, we describe in detail the case of regular spherical surgery. The almost spherical surgery is very similar.

First, we briefly review the relevant surgery constructions. We follow the scheme given by M. Walsh in great detail, see [37] and also [38].

12.1. Gromov-Lawson surgery.

Let M be a closed manifold, $\dim M = n-1$, and $S^p \times D^{q+1} \subset M$ be a sphere embedded with a trivial normal bundle, $p+q+1 = n-1$. Let M' be the manifold obtained as the result of surgery on M along S^p:

$$M' = (M \setminus (S^p \times D^{q+1})) \cup_{S^p \times S^q} D^{p+1} \times S^q. \tag{12.1}$$

We denote $I_0 = [0,1]$. It is convenient to attach the handle $D^{p+1} \times D^{q+1}$ to the cylinder $(M \times I_0)$ to obtain the cobordism V, the trace of the surgery between the manifolds M and M':

$$V = (M \times I_0) \cup_{(S^p \times D^{q+1}) \times \{1\}} D^{p+1} \times D^{q+1}, \quad \partial V = M \sqcup -M'.$$

Let g be a psc-metric on M. We assume that the codimension of the surgery is at least three, i.e., $q \geq 2$. The Gromov-Lawson procedure can be "formalized" as follows. The key step of the Gromov-Lawson construction is to deform the metric g near the sphere S^p to the standard metric. This could be done in two standard steps in order to modify the manifold $M \times I_0$, then construct a trace V of this surgery:

(1) We attach the cylinder $S^p \times D^{q+1} \times I_1$ to $M \times I_0$, where we identify

$$S^p \times D^{q+1} \times \{0\} \quad \subset \quad \partial(S^p \times D^{q+1} \times I_1) \quad \text{with}$$

$$S^p \times D^{q+1} \quad \subset \quad M \times \{1\} \subset \partial S^p \times D^{q+1}.$$

According to the Gromov-Lawson deformations, the metric g could be assumed to be already standard near the boundary $S^p \times D^{q+1} \times \{1\}$ of the $S^p \times D^{q+1} \times I_1$, i.e., $h_0 + ds^2$, where h_0 is a round metric on S^p and ds^2 is a flat metric on D^{q+1}.

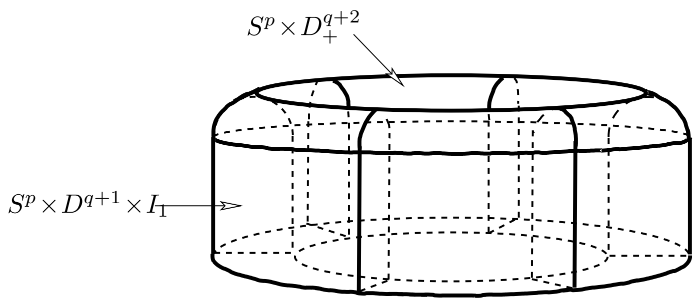

FIGURE 13. The part $(S^p \times D^{q+1} \times I_1) \cup (S^p \times D_+^{q+2})$ with torpedo metric.

(2) Next, let D_+^{q+2} be a half of the standard disk in $\mathbf{R}^{q+2}$; in particular, the boundary $\partial D_+^{q+2} = D^{q+1} \cup S_+^{q+1}$. We assume that D_+^{q+2} is equipped with a *standard torpedo metric*, as it is shown at Fig. 13.

Then we attach $S^p \times D_+^{q+2}$ to

$$M \times I_0 \cup S^p \times D^{q+1} \times I_1$$

by identifying

$$S^p \times D^{q+1} \times \{1\} \quad \subset \quad \partial(S^p \times D^{q+1} \times I_1) \quad \text{with}$$

$$S^p \times D^{q+1} \quad \subset \quad \partial(S^p \times D_+^{q+2}), \quad \text{(see Fig. 13).}$$

We denote by V_0 the resulting manifold:

$$V_0 = (M \times I_0) \cup (S^p \times D^{q+1} \times I_1) \cup (S^p \times D_+^{q+2}), \quad \text{(see Fig. 14, (a)).}$$

(3) To obtain a trace V of this surgery, we delete the "cup" $S^p \times D_+^{q+2}$ out of V_0 and attach the handle $D^{p+1} \times D^{q+1}$ by identifying

$$S^p \times D^{q+1} \times \{1\} \quad \subset \quad \partial(S^p \times D^{q+1} \times I_1) \quad \text{with}$$

$$S^p \times D^{q+1} \quad \subset \quad \partial(D^{p+1} \times D^{q+1}), \quad \text{(see Fig. 14, (b)).}$$

Thus for a psc-metric g on M, there is a "canonical" psc-metric $\tilde{g}$ on V, such that $\tilde{g}$ is a product-metric near the boundary:

$$\begin{aligned} \tilde{g} &= g + ds^2 \quad \text{near } M, \text{ and} \\ \tilde{g} &= g' + ds^2 \quad \text{near } M', \text{ with } R_{g'} > 0. \end{aligned} \tag{12.2}$$

Here s is a normal coordinate near the boundary of V.

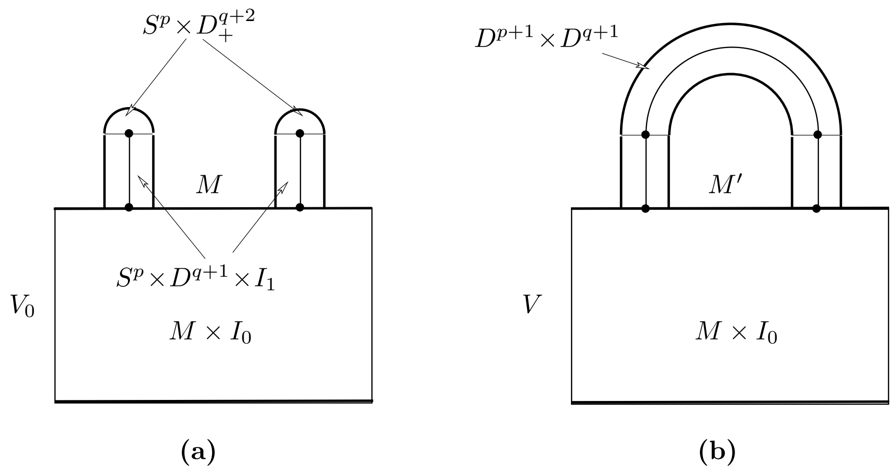

FIGURE 14. Trace of the surgery V between M and M'.

12.2. **Surgery and psc-isotopy.** This is easy. Let g_t be a smooth family of psc-metrics on M, $t \in [0,1]$. Then the above construction of the metric $\tilde{g}$ on the manifold V satisfying (12.2) depends smoothly on the metric g. Thus, we obtain a family of Riemannian manifolds $(V, \tilde{g}_t)$ such that the restriction $g'_t = \tilde{g}_t|_{M'}$ provides psc-isotopy between g'_0 and g'_1.

12.3. **Surgery and psc-concordance.** Let g_0 and g_1 be two psc-metrics on M. Then we have constructed the Riemannian manifolds $(V, \tilde{g}_0)$ and $(V, \tilde{g}_1)$ as above.

Now we assume that $(M \times [0,1], \bar{g})$ is a psc-concordance between psc-metrics g_0 and g_1. In particular, we assume that we are given $\varepsilon > 0$ such that

$$\bar{g}|_{M \times [0,\varepsilon)} = g_0 + dt^2, \quad \bar{g}|_{M \times (1-\varepsilon,1]} = g_1 + dt^2.$$

Furthemore, we assume that the metric $\bar{g}$ restricted to the strip $S^p \times D^{q+1} \times [0,1]$ is standard, i.e.,

$$\bar{g}|_{S^p \times D^{q+1} \times [0,1]} = g_{st}^{(p)} + g_{\mathrm{torp}}^{(q+1)} + dt^2. \tag{12.3}$$

Now we would like to extend the psc-concordance $(M \times [0,1], \bar{g})$ to a longer cylinder. We choose $a > 0$ and attach the cylinders $(M \times [-a, 0], g_0 + dt^2)$ and $(M \times [1, 1+a], g_1 + dt^2)$ to the psc-concordance $(M \times I, \bar{g})$:

$$M \times [-a, 1+a] = (M \times [-a, 0]) \cup (M \times [0,1]) \cup (M \times [1, 1+a]).$$

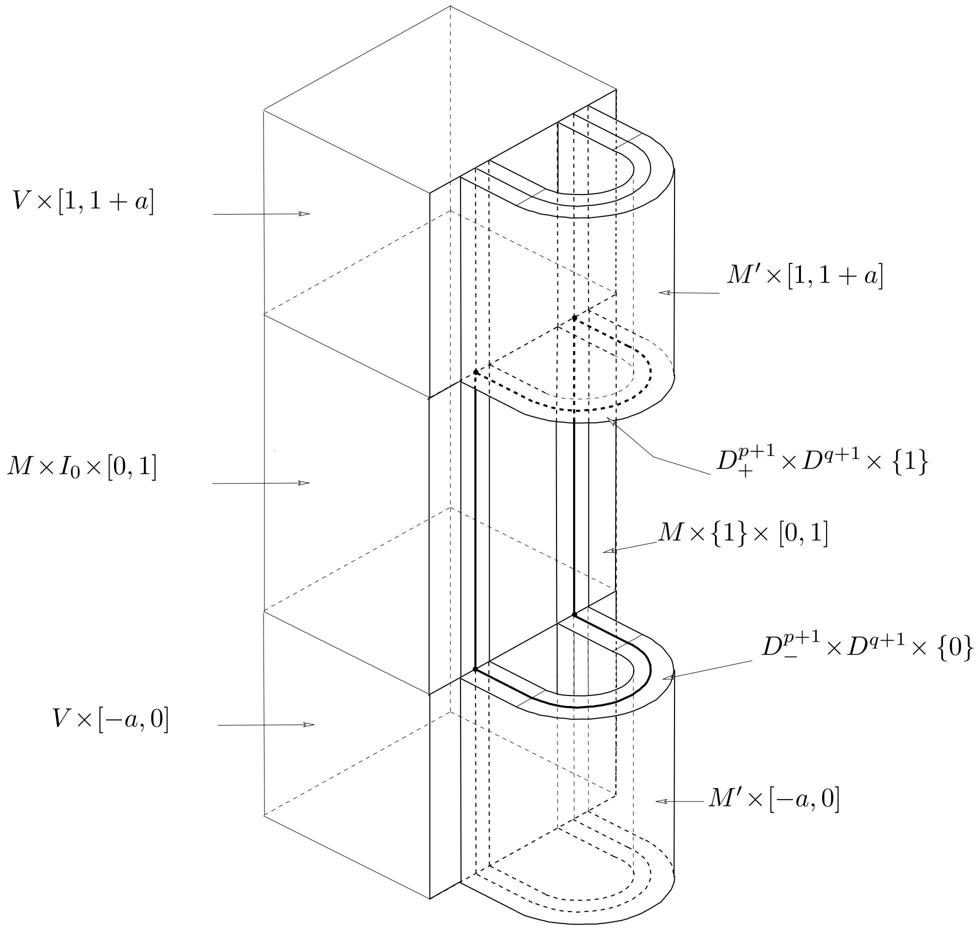

FIGURE 15. The manifold W_0: the first step to construct concordance between g_0' and g_1'.

We obtain the Riemannian manifold $(M \times [-a, 1+a], \hat{g})$, where

$$\hat{g}|_{M\times[-a,0]} = g_0 + dt^2, \quad \hat{g}|_{M\times I} = \bar{g}, \quad \hat{g}|_{M\times[1,1+a]} = g_1 + dt^2.$$

Now we construct the manifold W_0 as follows, see Fig. 15.

$$\begin{array}{lll} M \times I_0 \times \{0\} & \subset & V \times \{0\} \subset V \times [-a, 0] \quad \text{with} \\ \\ M \times I_0 \times \{0\} & \subset & M \times I_0 \times [0,1], \quad \text{and} \\ \\ M \times I_0 \times \{1\} & \subset & M \times I_0 \times [0,1], \quad \text{with} \\ \\ M \times I_0 \times \{1\} & \subset & V \times \{1\} \subset V \times [1, 1+a]. \end{array}$$

We glue together the manifolds $V\times[-a,0]$, $(M\times I_0)\times[0,1]$ and $V\times[1,1+a]$ as it is shown at Fig 15, i.e. we identify

$$M\times I_0\times\{0\} \quad\subset\quad V\times\{0\}\subset V\times[-a,0] \quad \text{with}$$

$$M\times I_0\times\{0\} \quad\subset\quad M\times I_0\times[0,1],$$

and also

$$M\times I_0\times\{1\} \quad\subset\quad M\times I_0\times[0,1], \quad \text{with}$$

$$M\times I_0\times\{1\} \quad\subset\quad V\times\{1\}\subset V\times[1,1+a].$$

We notice that the boundary of the manifold W_0 is decomposed as follows:

$$\partial W_0\cong(V\times\{-a\})\cup(M\times[-a,a+1])\cup(V\times\{1+a\})\cup Y',$$

where the manifold Y' is given as

$$\begin{aligned} Y' \quad = \quad & (M'\times[-a,0])\cup(D^{p+1}\times D^{q+1}\times\{0\})\cup \\ & (M\times\{1\}\times[0,1])\cup(D^{p+1}\times D^{q+1}\times\{1\})\cup(M'\times[1,1+a]), \end{aligned}$$

as it is shown in Fig. 15. According to the assumption (12.3), we have that the psc-concordance $\bar{g}$ on $M\times I$ and its extension, the metric $\hat{g}$ to $M\times[-a,1+a]$, are standard on the strip

$$S^p\times D^{q+1}\times[-a,1+a].$$

Thus we obtain a psc-metric G_0 on W_0, and can assume that G_0 is a product metric on the submanifold $M\times I_0\times[-a,1+a]$ and standard on the handles

$$(D^{p+1}\times D^{q+1})\times[-a,0] \;\text{ and }\; (D^{p+1}\times D^{q+1})\times[1,1+a]).$$

We would like to perform the second surgery, this time on the manifold Y', in such a way that the resulting manifold will be diffeomorphic to the cylinder $M'\times[-a,1+a]$. We notice that we are given a canonical embedding $S^{p+1}\times D^{q+1}\subset Y'$. Here, the sphere S^{p+1} is decomposed as follows:

$$S^{p+1}=(D^{p+1}_-\times\{0\})\cup(S^p\times[0,1])\cup(D^{p+1}_+\times\{1\})$$

(see Fig. 16). Clearly the induced metric h on S^{p+1} is not standard; however, after smoothing corners, the metric h on S^{p+1} is given by stretching and "bending twice" the standard metric (see Fig. 16). Next, in order to turn the metric on $S^{p+1}\times D^{q+1}$ into standard, torpedo metric, we attach the cylinder $S^{p+1}\times D^{q+1}\times I_1$ and after that the handle $S^{p+1}\times D^{q+2}_+$ with the "topedo" metric as it is shown in Fig 17. Here $\partial D^{q+2}_+=D^{q+1}\cup S^{q+1}_+$, where S^{q+1}_+ is a hemisphere equipped with a

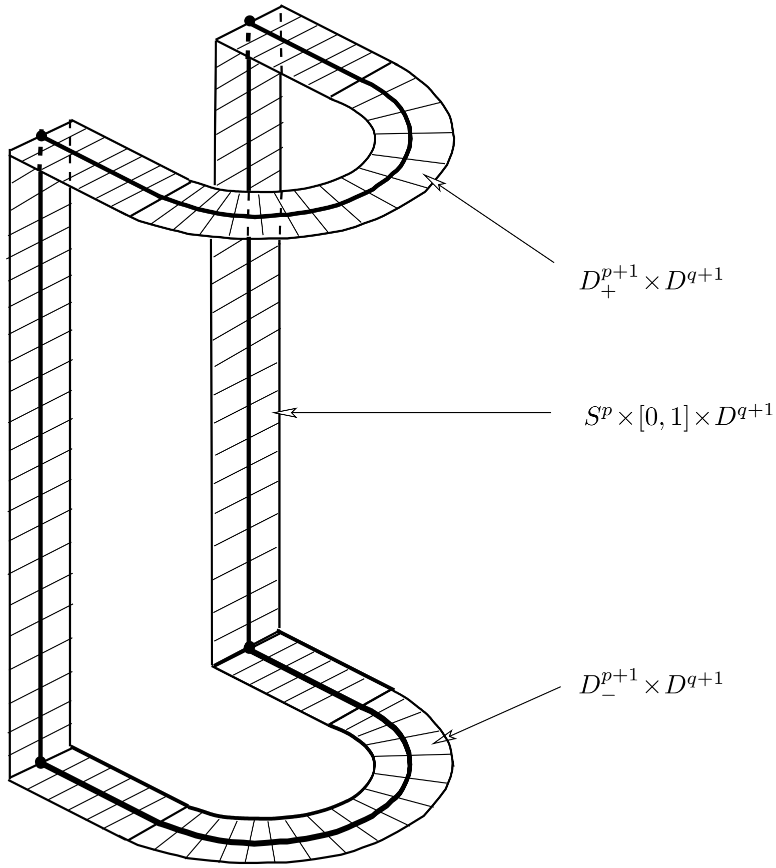

FIGURE 16. The embedding $S^{p+1} \times D^{q+1}$ to Y'

torpedo metric. We identify:

$$S^{p+1} \times D^{q+1} \times \{0\} \subset S^{p+1} \times D^{q+1} \times I_1 \quad \text{with}$$

$$S^{p+1} \times D^{q+1} \subset Y; \quad \text{and then}$$

$$S^{p+1} \times D^{q+1} \subset S^{p+1} \times (D^{q+1} \cup S^{q+1}_+) = \partial D^{q+2}_+ \quad \text{with}$$

$$S^{p+1} \times D^{q+1} \times \{1\} \subset S^{p+1} \times D^{q+1} \times I_1,$$

(see Fig. 17). The resulting manifold W_1 is "surgery-ready".

To perform the surgery, we just delete the manifold $S^{p+1} \times D^{q+2}_+$ and instead attach the handle $D^{p+2} \times D^{q+1}$ to W_1 by identifying the manifolds

$$S^{p+1} \times D^{q+1} \times \{1\} \subset S^{p+1} \times D^{q+1} \times I_1 \quad \text{with}$$

$$S^{p+1} \times D^{q+1} \subset \partial(D^{p+2} \times D^{q+1}) \subset D^{p+2} \times D^{q+1}.$$

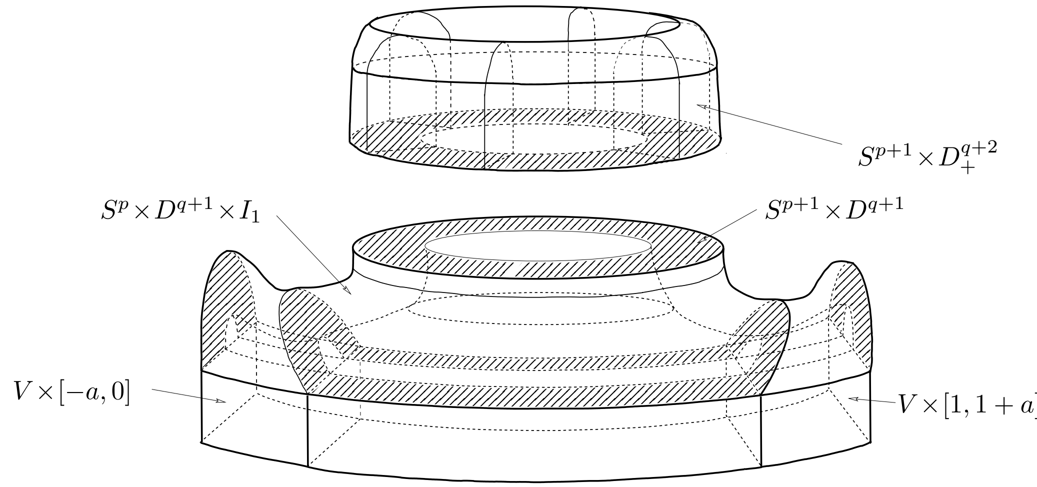

FIGURE 17. Preparation for the second surgery

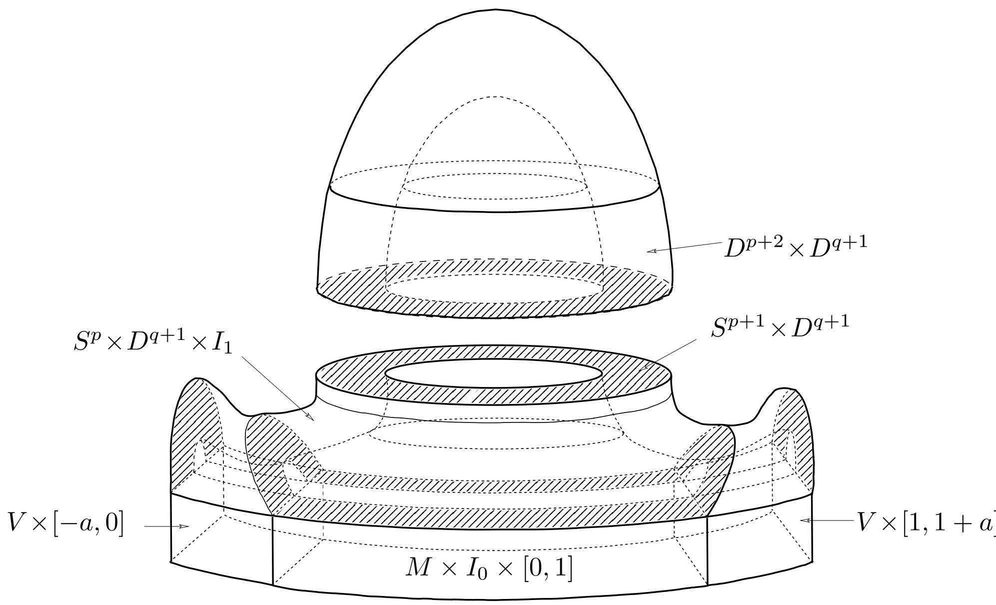

FIGURE 18. The second surgery

We denote by W the resulting manifold: $W = W_1 \cup D^{p+2} \times D^{q+1}$ (see Fig 18). One can easily see that W is diffeomorphic to $V \times [-a, 1+a]$, in particular,

$$\partial W \cong (V \times \{-a\}) \cup (M \times [-a, 1+a]) \cup (V \times \{1+a\}) \cup (M' \times [-a, 1+a]),$$

and the manifold $M' \times [-a, 1+a]$ is given a psc-metric $\bar{g}'$, so that the Riemannian manifold

$$(M' \times [-a, 1+a], \bar{g}')$$

is a psc-concordance joining the psc-metrics g_0' and g_1' on M'. This shows that if g_0, g_1 are psc-concordant psc-metrics on M, then the Gromov-Lawson construction yields psc-concordant psc-metrics g_0', g_1' on M'.

12.4. **Surgery and pseudo-isotopies.** So far we have suppressed a role of pseudo-isotopies. Now we would like to explain how to deal with an arbitrary psc-concordance.

First, we have to recall Hudson's results, [20], and some relevant definitions. Let $N \subset M$ be a submanifold, where M is a closed smooth manifold. We denote $i_N : N \to M$ the inclusion map. Then a map $F : N \times I \to M \times I$ is called an *allowable concordance* if $F|_{N\times\{0\}} = i_N$ and $F^{-1}(M \times \{0\}) = N \times \{0\}$. Consider the product $M \times I$ as a manifold with the boundary

$$\partial(M \times I) = M \times \{0\} \sqcup -M \times \{1\}.$$

An isotopy H of $M \times I$ is a map

$$H : (M \times I) \times I \to (M \times I) \times I$$

such that $H((M \times I) \times \{t\}) = (M \times I) \times \{t\}$. We need the following result:

Theorem 12.1. (Hudson, [20, Theorem 2.1]) *Let M be a closed manifold, $N \subset M$ be a submanifold of codimesion at least three, and $F : N \times I \to M \times I$ be an allowable concordance. Then there exists an isotopy $H : (M \times I) \times I \to (M \times I) \times I$ fixed on $M \times \{0\}$ such that the composition*

$$N \times I \times \{1\} \xrightarrow{F} M \times I \xrightarrow{H|_{M\times I\times\{1\}}} M \times I$$

coincides with the map $i_N \times Id_I : N \times I \to M \times I$, where $i_N : N \hookrightarrow M$ is the inclusion map.

Remark. We stated Theorem 12.1 in the relevant terms we use in this paper. In [20], Hudson proves more general results for manifolds with boundary in both, smooth and piece-wise linear cases. We also note that the restriction on the codimension is crucial for us: this matches perfectly with the condition our requirement that a surgery has to be admissible.

Now let $g_0, g_1 \in \mathcal{R}\text{iem}^+(M)$ such that there exists a pseudo-isotopy $\bar{\varphi} : M \times I \to M \times I$ so that the metrics g_0 and $g_1^* := (\bar{\varphi}|_{M\times\{1\}})^* g_1$ are psc-isotopic.

We fix a surgery sphere $S^p \subset M \times \{0\}$ embedded to the base $M \times \{0\}$ of $M \times I$. We assume that the sphere S^p has trivial normal bundle, and we choose an embedding $S^p \times D^{q+1} \subset M \times \{0\}$ of its tubular neighborhood, where a codimension of the sphere S^p in M is $q + 1 \geq 3$.

We denote by F the restriction $\bar{\varphi}|_{S^p\times I}$. Clearly, the map $F : S^p \times I \to M \times I$ is an allowable concordance. Since the codimension of S^p in M is at least three, we use Theorem 12.1 to find an isotopy $H : (M \times I) \times I \to (M \times I) \times I$ fixed on $M \times \{0\}$ such that the composition

$$S^p \times I \times \{1\} \xrightarrow{F} M \times I \xrightarrow{H|_{M\times I\times\{1\}}} M \times I \tag{12.4}$$

coincides with the map $i_{S^p} \times Id_I : S^p \times I \to M \times I$, where $i_{S^p} : S^p \hookrightarrow M$ is the inclusion map.

We denote by $\tilde{F} = \bar{\varphi}|_{(S^p\times D^{q+1})\times I}$ the restriction of the concordance $\bar{\varphi}$ to $(S^p \times D^{q+1}) \times I$. Since the sphere S^p has a trivial normal bundle, we can assume that the isotopy H is chosen in

such a way that (12.4) extends to a tubular neighborhood $S^p \times D^{q+1}$, i.e., the following diagram commutes:

$$\begin{tikzcd}
S^p \times I \times \{1\} \arrow[r, "F"] \arrow[d, "i_0"] & M \times I \arrow[r, "H|_{M\times I\times\{1\}}"] \arrow[d, "Id_{M\times I}"] & M \times I \arrow[d, "Id_{M\times I}"] \\
(S^p \times D^{q+1}) \times I \times \{1\} \arrow[r, "\tilde{F}"] & M \times I \arrow[r, "H|_{M\times I\times\{1\}}"] & M \times I
\end{tikzcd},$$

where $i_0 : (S^p \times \{0\}) \times I \times \{1\} \to (S^p \times D^{q+1}) \times I \times \{1\}$ is the zero-section. Furthemore, by standard arguments, we can assume that the composition

$$(S^p \times D^{q+1}) \times I \times \{1\} \xrightarrow{F} M \times I \xrightarrow{H|_{M\times I\times\{1\}}} M \times I \tag{12.5}$$

coincides with the map $i_{S^p\times D^{q+1}} \times Id_I : (S^p \times D^{q+1}) \times I \to M \times I$.

Thus we can deform the concordance $\bar{\varphi} : M \times I \to M \times I$ by means of the isotopy H so that the composition

$$\bar{\psi} : M \times I \xrightarrow{\bar{\varphi}} M \times I \xrightarrow{H|_{M\times I\times\{1\}}} M \times I$$

is a pseudo-isotopy that restricts to the identity map on the cylinder $S^p \times D^{q+1} \times I$.

Consider the cylinder $M \times I$. Then by construction, the metrics $g_1^* := (\bar{\varphi}|_{M\times\{1\}})^* g_1$ and $g_1^{**} := (\bar{\psi}|_{M\times\{1\}})^* g_1$ are psc-isotopic. However, the metric g_1^{**} coincides with the original metric g_1 on the neighbourhood $S^p \times D^{q+1} \times \{1\} \subset M \times \{1\}$ of the surgery sphere. We choose a psc-isotopy g_t^{**} between g_0 and g_1^{**} such that the metric $\bar{g}^{**} = g_t^{**} + dt^2$ on $M \times I$ has positive scalar curvature $R_{\bar{g}^{**}} > 0$ and it is a product-metric near the boundary $M \times \{i\}$, $i = 0, 1$.

Now we consider the manifold with corners $V_0 \times I = M \times I \times I$, where the face $M \times \{0\} \times I$ is identified with the above psc-concordance $(M \times I, \bar{g}^{**})$. We need the following result:

Proposition 12.2. *There exists a metric $\bar{\mathbf{g}}^{**}$ on $V_0 \times I$ such that*

(i) $R_{\bar{\mathbf{g}}^{**}} > 0$ *everywhere on* $V_0 \times I$*;*
(ii) *the restriction $\bar{\mathbf{g}}^{**}$ to the face $M \times \{0\} \times I$ coincides with $\bar{g}^{**}$;*
(iii) *the restriction $\bar{\mathbf{g}}^{**}$ to the face $M \times \{1\} \times I$ is surgery-ready, i.e., it coincides with the metric $g_o^{(p)} + g_{\mathrm{torp}}^{(q+1)}(\varepsilon) + dt^2$ on the strip*
$$S^p \times D^{q+1} \times I \subset M \times \{1\} \times I;$$
(iv) *the metric $\bar{\mathbf{g}}^{**}$ is a product-metric near all faces of $M \times I \times I$.*

A proof of Proposition 12.2 requires slight modification of the technique developed in [37, 38]; however, we leave this proof to an interested reader as an exercise.

In particular, we have that the metrics $\tilde{g}_0 := \bar{\mathbf{g}}^{**}|_{M\times\{0\}\times\{1\}}$ and g_0 are psc-isotopic, as well as the metrics $\tilde{g}_1^{**} := \bar{\mathbf{g}}^{**}|_{M\times\{1\}\times\{1\}}$ and g_1^{**} are psc-isotopic. We use (iii) from Proposition 12.2 and the procedure described in Section 12.3 to perform a Gromov-Lawson surgery on the sphere S^p

along the strip

$$S^p \times D^{q+1} \times I \subset M \times \{1\} \times I$$

to obtain psc-metrics g_0' and $(\tilde{g}_1^{**})'$ on the manifold M' together with psc-isotopy between the psc-metrics g_0' and $(\tilde{g}_1^{**})'$.

We note that the metric g_0' is nothing but the original Gromov-Lawson metric on M' by construction. Again, by construction, the metric $(\tilde{g}_1^{**})'$ is related to the original psc-metric g_1' via the pseudo-isotopy

$$\bar{\psi}' : M' \times I \to M' \times I$$

which extends the pseudo-isotopy $\bar{\psi} : M \times I \to M \times I$ by the identity on the attached handle $(D^{p+1} \times S^q) \times I \subset M' \times I$.

We conclude that if psc-metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ are psc-isotopic up to pseudo-isotopy on M, then the psc-metrics $g_0', g_1' \in \mathcal{R}\mathrm{iem}^+(M')$ obtained from the metrics g_0, g_1 by means of the Gromov-Lawson surgery, are psc-isotopic up to pseudoisotopy on M'. This completes the proof of Theorem 2.3. $\square$

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